# Non-trapping condition for semiclassical Schrödinger operators with matrix-valued potentials.

Thierry Jecko

*À Kiliann, mon fils.*

# Condition de non-capture pour des opérateurs de Schrödinger semi-classiques à potentiels matriciels.

Thierry Jecko
IRMAR, Université de Rennes I,
Campus Beaulieu,
F-35042 Rennes Cédex, France
thierry.jecko@univ-rennes1.fr
12-03-04

**Résumé**

On considère des opérateurs de Schrödinger semi-classiques à potentiels matriciels, lisses et de longue portée, dont différentes valeurs propres peuvent se croiser sur une sous-variété de codimension une. On note par $h$ le paramètre semi-classique et on s'intéresse aux énergies strictement supérieures au bas du spectre essentiel. Sous une certaine condition d'invariance le long du croisement, portant sur la structure matricielle du potentiel, et sous une certaine condition de structure à l'infini, on démontre que les valeurs aux bords de la résolvante à l'énergie $\lambda$, vues comme opérateurs bornés sur des espaces à poids convenables, sont $O(h^{-1})$ *si et seulement si* $\lambda$ est une énergie non-captive pour le flot hamiltonien généré par chaque valeur propre du symbole de l'opérateur.

**Mots clé :** Condition de non-capture, croisement de valeurs propres, opérateur de Schrödinger matriciel, théorie de Mourre, estimations semi-classiques de résolvante, états cohérents, théorème d'Egorov, mesure semi-classique.

# Non-trapping condition for semiclassical Schrödinger operators with matrix-valued potentials.

Thierry Jecko
IRMAR, Université de Rennes I,
Campus Beaulieu,
F-35042 Rennes Cédex, France
thierry.jecko@univ-rennes1.fr
12-03-04

**Abstract**

We consider semiclassical Schrödinger operators with matrix-valued, long-range, smooth potential, for which different eigenvalues may cross on a codimension one submanifold. We denote by $h$ the semiclassical parameter and we consider energies above the bottom of the essential spectrum. Under some invariance condition on the matricial structure of the potential near the eigenvalues crossing and some structure condition at infinity, we prove that the boundary values of the resolvent at energy $\lambda$, as bounded operators on suitable weighted spaces, are $O(h^{-1})$ *if and only if* $\lambda$ is a non-trapping energy for all the Hamilton flows generated by the eigenvalues of the operator's symbol.

**Keywords:** Non-trapping condition, eigenvalues crossing, Schrödinger matrix operators, Mourre theory, semiclassical resolvent estimates, coherent states, Egorov's theorem, semiclassical measure.

# Contents


**1 Introduction and main results.** **4**

**2 Detailed assumptions on the models.** **9**

2.1 General case. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9

2.2 Codimension 1 crossings. . . . . . . . . . . . . . . . . . . . . . . . . 12

**3 Semiclassical trapping.** **15**

3.1 Generalization of scalar results. . . . . . . . . . . . . . . . . . . . . 15

3.2 A matricial difficulty. . . . . . . . . . . . . . . . . . . . . . . . . . 18

3.3 Codimension 1 crossings with an invariance condition. . . . . . . . . . . 20

**4 Toward the non-trapping condition.** **22**

4.1 Wang's strategy. . . . . . . . . . . . . . . . . . . . . . . . . . . . . 22

4.2 A Gronwall type argument. . . . . . . . . . . . . . . . . . . . . . . . 24

**5 Previous results revisited.** **27**

# 1 Introduction and main results.

In the scattering theory of Schrödinger operators, several semiclassical results, with respect to the "Planck constant", are based on semiclassical resolvent estimates (as a typical example, see [RT]). If one seeks for similar results in the semiclassical framework of the Born-Oppenheimer approximation for molecular Schrödinger operators, it is interesting to generalize these semiclassical resolvent estimates to Schrödinger operators with operator-valued potential (see [J2, J3]). On this way, it is worth to treat the case of matrix-valued potentials (see [KMW, J1]). Among them the radial ones are also of interest since they correspond to diatomic molecular operators.
In order to present this project in detail, we need some notation. Taking $m \in \mathbb{N}^*$, let $\mathcal{M}_m(\mathbb{C})$ be the algebra of $m \times m$ matrices with complex coefficients, endowed with the operator norm denoted by $\| \cdot \|_m$. We denote by $\mathrm{I}_m$ the corresponding identity matrix. Let $d \in \mathbb{N}^*$ and let $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$ be the space of $\mathbb{C}^m$-valued $\mathrm{L}^2$ functions on $\mathbb{R}^d$, equiped with its usual norm $\| \cdot \|$ and scalar product $\langle \cdot, \cdot \rangle$. We denote by $\Delta_x$ the Laplacian in $\mathbb{R}^d$. The semiclassical parameter is $h \in ]0; h_0]$, for some $h_0 > 0$. The semiclassical Schrödinger operator we consider is the unbounded operator

$$\hat{P}(h) \ := \ -h^2 \Delta_x \mathrm{I}_m + M(x) \,, \tag{1.1}$$

acting in $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$, where $M(x)$ is the multiplication operator by a self-adjoint matrix $M(x) \in \mathcal{M}_m(\mathbb{C})$. We require that $M$ is smooth on $\mathbb{R}^d$ and tends, as $|x| \to \infty$, to some matrix $M_\infty$, which is of course self-adjoint. Furthermore we demand that the potential $M(x) - M_\infty$ is long-range (see (2.1)). It is well known that, under this assumption on $M$, the operator $\hat{P}(h)$ is self-adjoint on the domain of $\Delta_x \mathrm{I}_m$ (see [RS2] for instance). Its resolvent will be denoted by $R(z) := (\hat{P}(h) - z)^{-1}$, for $z$ in the resolvent set of $\hat{P}(h)$. Setting $\langle x \rangle := (1 + |x|^2)^{1/2}$, we denote by $\mathrm{L}^{2,s}(\mathbb{R}^d; \mathbb{C}^m)$, for $s \in \mathbb{R}$, the weighted $\mathrm{L}^2$ space of mesurable, $\mathbb{C}^m$-valued functions $f$ on $\mathbb{R}^d$ such that $x \mapsto \langle x \rangle^s f(x)$ belongs to $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$. It follows from Mourre theory ([M, CFKS]), with the dilation generator as scalar conjugate operator, that the resolvent has boundary values $R(\lambda \pm i0)$, as bounded operators from $\mathrm{L}^{2,s}(\mathbb{R}^d; \mathbb{C}^m)$ to $\mathrm{L}^{2,-s}(\mathbb{R}^d; \mathbb{C}^m)$ for any $s > 1/2$, provided that $\lambda$ is outside the pure point spectrum of $\hat{P}(h)$ and above the operator norm $\|M_\infty\|_m$ of the matrix $M_\infty$.
A careful inspection of the paper by Froese and Herbst (cf. [FH, CFKS]) shows that its result, namely the absence of eigenvalues above the bottom of the essential spectrum, extends to the present situation. This means that the boundary values $R(\lambda \pm i0)$ are well defined for any $\lambda > \|M_\infty\|_m$. It turns out that we do not need this fact here (see Remark 3.1). We choose to forget it in the formulation of our results. This could be useful for a possible generalization of our results to other operators, for which eigenvalues may be embedded in the essential spectrum.
The operator $\hat{P}(h)$ in (1.1) can be viewed as a $h$-pseudodifferential operator obtained by Weyl $h$-quantization (see (3.2)) of the symbol $P$ defined by

$$\forall \, \mathrm{x}^* := (x, \xi) \in T^*\mathbb{R}^d \,, \quad P(\mathrm{x}^*) \ := \ |\xi|^2 \mathrm{I}_m \, + \, M(x) \,, \tag{1.2}$$

with self-adjoint values in $\mathcal{M}_m(\mathbb{C})$. Notice that this symbol does not depend on the kind of $h$-quantization (cf. [Ba, R]).

Let $\lambda > \|M_\infty\|_m$. Our aim in this paper is to characterize, in terms of the energy $\lambda$ and of the symbol $P$, the following property: for all $s > 1/2$, there is an open interval $I$ about $\lambda$ and $h_0 > 0$ such that, for all $\mu \in I$, for all $h \in ]0; h_0]$,

$$\text{there exists} \quad \lim_{\epsilon \to 0^+} \langle x\rangle^{-s} R(\mu + i\epsilon)\langle x\rangle^{-s} \;=:\; \langle x\rangle^{-s} R(\mu + i0)\langle x\rangle^{-s} \tag{1.3}$$

$$\text{and } \exists C_{s,I} > 0\,;\, \forall \mu \in I\,;\, \forall h \in ]0; h_0]\,, \quad \left\|\langle x\rangle^{-s} R(\mu + i0)\langle x\rangle^{-s}\right\| \;\leq\; C_{s,I} \cdot h^{-1}\,. \tag{1.4}$$

Here $\|\cdot\|$ denotes the operator norm of the bounded operators on $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$. For short, we call this property "the property ((1.3) and (1.4))", which covers the semiclassical resolvent estimates mentionned above.

In the scalar case, a characterization is well-known (see [RT, GM, W, VZ, B, J5]). Let us describe it. Let $q \in C^\infty(T^*\mathbb{R}^d; \mathbb{R})$ be a Hamilton function and denote by $\phi^t$ its Hamilton flow. An energy $\mu$ is non-trapping for $q$ (or for the flow $\phi^t$) if

$$\forall \mathrm{x}^* \in q^{-1}(\mu), \quad \lim_{t \to -\infty} |\phi^t(\mathrm{x}^*)| \;=\; +\infty \quad \text{and} \quad \lim_{t \to +\infty} |\phi^t(\mathrm{x}^*)| \;=\; +\infty\,. \tag{1.5}$$

For $m = 1$, the property ((1.3) and (1.4)) at energy $\lambda$ and the same property applied to $R(\mu - i\epsilon)$ instead of $R(\mu + i\epsilon)$ hold true if and only if $\lambda$ is a non-trapping energy for the symbol $P$ (see (1.2)).

In the matricial case, the usual definition of the Hamilton flow for a matrix-valued Hamilton function does not make sense. But we hope that there is a characterization on the symbol $P$. From [J1], we learn that, if the multiplicity of the eigenvalues of $P$ is constant, then the non-trapping condition at energy $\lambda$ for each eigenvalue (see (1.5)) is sufficient (and even necessary, cf. Theorem 1.8) to get the property ((1.3) and (1.4)) at energy $\lambda$. The general case is however more complicated since one can find a smooth matrix-valued potential $M$ such that this non-trapping condition does not make sense, since the eigenvalues are not enough regular (cf. [K]).

In this paper, we choose to look at the case where the eigenvalues and the corresponding eigenprojections are smooth and where a (simple) crossing of eigenvalues occurs. This situation already contains interesting potentials. Since our result in [J4] in this situation is rather limited and unsatisfactory (see also Section 5), we think it is worth to better understand this case before treating the general one.

Let us now present our main results. They concern the model of codimension 1 crossings, for which the eigenvalues crossing is locally a codimension 1 crossing in the sense given in [H]. Details are provided in Subsection 2.2. We obtain Theorem 1.2 if, roughly speaking, the eigenprojections of $M$ do not varry along the crossing and we derive Theorem 1.3 if, roughly speaking, the "crossing at infinity" has the structure of a codimension 1 crossing. These results yield together the desired characterization for potentials $M$ satisfying the two conditions.

**Theorem 1.1.** *Consider the model of codimension* 1 *crossings* (*see Definition 2.6*) *that satisfies the invariance condition at the crossing* (*see Definition 2.6*) *and the structure condition at infinity* (*see Definition 2.9*). *Let* $\lambda > \|M_\infty\|_m$. *The property* ((1.3) *and* (1.4)) *holds true if and only if* $\lambda$ *is non-trapping for all eigenvalues of the symbol* $P$ *of* $\hat{P}(h)$ (*cf.* (1.5) *and Definition 2.6*).

**Proof:** While the "if" part follows from Theorem 1.2, the "only if" part is a consequence of Theorem 1.3. □

**Theorem 1.2.** *Consider the model of codimension* 1 *crossings* (*see Definition 2.6*) *that satisfies the invariance condition at the crossing* (*see Definition 2.6*). *Let* $\lambda > \|M_\infty\|_m$. *If* $\lambda$ *is non-trapping for all eigenvalues of the symbol* $P$ *of* $\hat{P}(h)$ (*cf.* (1.5) *and Definition 2.6*), *then the property* ((1.3) *and* (1.4)) *holds true.*

**Proof:** See the end of Subsection 3.3. □

**Theorem 1.3.** *Consider the model of codimension* 1 *crossings* (*see Definition 2.6*) *that satisfies the structure condition at infinity* (*see Definition 2.9*). *Let* $\lambda > \|M_\infty\|_m$. *If the property* ((1.3) *and* (1.4)) *holds true then* $\lambda$ *is non-trapping for all eigenvalues of the symbol* $P$ *of* $\hat{P}(h)$ (*cf.* (1.5) *and Definition 2.6*).

**Proof:** See the end of Subsection 4.2. □

**Remark 1.4.** *Theorem 1.2 applies for a model of codimension* 1 *crossings with a radial potential* $M$, *that is when* $M$ *is a function of* $|x|$ (*cf. Remark 2.7*). *If there is "no crossing at infinity"* (*see Definition 2.8*), *then the structure condition at infinity is automatically satisfied* (*cf. Remark 2.10*) *and Theorem 1.3 applies with a simplier proof. We point out that, for Theorems 1.2 and 1.3, we need the structure assumptions from Definitions 2.6 and 2.9, only "below" the considered energy* $\lambda$ (*cf. Remark 2.11*).

**Remark 1.5.** *Theorems 1.2 and 1.3 hold true (and, thus, so does Theorem 1.1) if we replace* $R(\mu + i\epsilon)$ *by* $R(\mu - i\epsilon)$ *in* (1.3) *and* (1.4). *Notice that, for* $m = 1$, *Theorem 1.3 is slightly stronger than the corresponding result in [W]. Indeed, we only need estimates on one boundary value of the resolvent, thanks to Proposition 1.7 below.*

Although we decided to focus on codimension 1 crossings, we derive some partial results concerning the general case in Subsections 2.1, 3.1, 3.2, and in Section 5. Among them, we point out the following two propositions. From Theorem 1.1, we see that the property ((1.3) and (1.4)) is in fact independent of $s > 1/2$ and of the sign chosen in the resolvent, if the non-trapping condition is satisfied. This holds true in a more general framework, as proved in Propositions 1.6 and 1.7 below.

**Proposition 1.6.** *Consider the general model* (*see Subsection 2.1*) *and let* $\lambda > \|M_\infty\|_m$, *such that, for* some $s > 1/2$, (1.3) *holds true near* $\lambda$ *and for* $h$ *small enough. Then, this holds true for* any $s > 1/2$. *If, for* some $s > 1/2$, *the resolvent estimate* (1.4) *is satisfied near* $\lambda$ *and for* $h$ *small enough, then this is true for* any $s > 1/2$.

**Proposition 1.7.** *Consider the general model* (*see Subsection 2.1*) *and let* $\lambda > \|M_\infty\|_m$. *If, for some* $s > 1/2$, (1.3) *and the resolvent estimate* (1.4) *hold true near* $\lambda$ *and for* $h$ *small enough, then, for any* $r > 1/2$, $\langle x\rangle^{-r} R(\mu - i0)\langle x\rangle^{-r}$ *is well defined for* $\mu$ *close enough to* $\lambda$ *and* $h$ *small enough. Furthermore,* (1.4) *for* $s$, *with* $R(\mu + i0)$ *replaced by* $R(\mu - i0)$, *holds true for* $\mu$ *near* $\lambda$ *and* $h$ *small enough.*

**Proof of Propositions 1.6 and 1.7:** see the end of Subsection 3.1. □

As announced above, we can complete the result in [J1] as follows.

**Theorem 1.8.** *Consider the general model (see Subsection 2.1), let $\lambda > \|M_\infty\|_m$, and assume that the relevant eigenvalues crossing at energy $\lambda$ (cf. Definition 2.2) is empty. Then the property ((1.3) and (1.4)) holds true if and only if $\lambda$ is non-trapping for all relevant eigenvalues of the symbol $P$ at energy $\lambda$ (cf. (1.5) and Definition 2.4).*

**Proof:** In [J1], the "if" part was proved and we shall see another proof in Section 5. The "only if" part follows from Propositions 1.7 and 4.1. □

Concerning the proof of our results, we want to add some comments. For the proof of Theorem 1.2 (see Section 3), we do not follow the semiclassical version of Mourre method, as in [J1, J4]. We prefer to use Burq's strategy (see [B]), that we adapted to the scalar, long-range scattering in [J5]. Most of the results obtained in [J5] are easily extended to the present situation, in the general case, and we also get Propositions 1.6 and 1.7. Notice that the first statement in Propositions 1.6 and 1.7 is a consequence of Mourre theory (cf. [M, CFKS]). Following [J5] further, we meet a matricial difficulty (in Subsection 3.2), based on the fact that matrices do not commute in general. For the model of codimension 1 crossings satisfying the invariance condition at the crossing, we can remove it and conclude as in [J5]. In Section 5, we compare this result with those in [J1, J4] and show how the later can be proved along the present lines.
To prove Theorem 1.3 (see Section 4), we follow the strategy in [W], which works in the scalar case. If the eigenvalues crossing is empty, it is easy to adapt it and get the "only if" part of Theorem 1.8. Since this strategy relies on a semiclassical Egorov's theorem, which is not available for general matrix operators (even for the model of codimension 1 crossings), we need an important modification of Wang's lines. Its main ingredient is a Gronwall type argument (see Subsection 4.2).
To end this introduction, let us connect our results to other questions treated in the litterature. If one considers empty eigenvalues crossings, we refer to [S] for scattering considerations and to [BG, BN] for the semiclassical Egorov's theorem. When eigenvalues cross, it is interesting to look at the case where resonances appear, as in [Né], since the resolvent estimates might be false in this case. Finally, it is worth to compare our results with those around the propagation of coherent states and the Landau-Zener formula. Among others, we refer to [CLP, FG, H]. Since the time is kept bounded and the evolution is measured strongly (not in norm), these results cannot imply ours but they do give an idea of what might happen in our framework. For instance, they do not reveal any capture phenomenon at the crossing, giving some hope to find a simple non-trapping condition in some cases. This actually motivated the present paper. We believe that we can learn more from these papers and also from [Ka], where some norm control on finite time evolution is given.

**Acknowledgment**:
the author thanks, for fruitful discussions and advices, V.Bach, V.Bruneau, F. Castella, Y. Colin de Verdière, C. Fermanian-Kamerer, C. Gérard, P. Gérard, G. Hagedorn, F. Hérau, A. Joye, M. Klein, A. Laptev, N. Lerner, A. Martinez, G. Métivier, F. Nier, and X.P. Wang.
This work was partially written at the institute Mittag-Leffler in Stockholm and at the university of Mainz, where the author was invited. He also thanks Ari Laptev and Volker Bach for these invitations.

# 2 Detailed assumptions on the models.

In this section, we describe precisely the considered models. We first introduce the general case. Then we present some class of codimension 1 eigenvalues crossings in the spirit of [H]. Finally, we explain the invariance condition at the crossing and the structure condition at infinity, that appear in Theorem 1.1.

## 2.1 General case.

In our general framework, we assume that the potential $M$ of the semiclassical Schrödinger operator $\hat{P}(h)$, given in(1.1), is a self-adjoint matrix-valued, smooth, long-range function on $\mathbb{R}^d$. This means that the values of $M$ belong to $\mathcal{M}_m(\mathbb{C})$ and are self-adjoint, that $M$ is $C^\infty$ on $\mathbb{R}^d$, and that there exist some $\rho > 0$ and some self-adjoint matrix $M_\infty$ such that

$$\forall\gamma \in \mathbb{N}^d\,,\ \forall x \in \mathbb{R}^d\,,\quad \Big\|\partial_x^\gamma\Big(M(x) - M_\infty\Big)\Big\|_m \;=\; O_\gamma\Big(\langle x\rangle^{-\rho-|\gamma|}\Big) \tag{2.1}$$

where $\|\cdot\|_m$ denotes the operator norm on $\mathcal{M}_m(\mathbb{C})$ and $\langle x\rangle = (1+|x|^2)^{1/2}$.
As already mentioned in Section 1, the operator $\hat{P}(h)$ is, if $M$ satisfies (2.1), self-adjoint on the domain of $\Delta_x \mathrm{I}_m$, which is isomorphic to the $m$th power of the domain of the $\Delta_x$ in $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C})$. Furthermore, $\hat{P}(h)$ may be viewed as the Weyl $h$-quantization (see (3.2)) of the symbol $P$ given by (1.2). Since we want to consider the resolvent of $\hat{P}(h)$ near some energy $\lambda \in \mathbb{R}$, we introduce smooth localization functions near $\lambda$. Precisely, for $\epsilon > 0$, let $\theta \in C_0^\infty(\mathbb{R};\mathbb{R})$ such that $\theta(\lambda) \neq 0$ and $\operatorname{supp}\theta \subset ]\lambda-\epsilon;\lambda+\epsilon[$. The operator $\theta(\hat{P}(h))$ localizes in energy for $\hat{P}(h)$ near $\lambda$. Furthermore, it is a $h$-pseudodifferential operator satisfying (3.3) (cf. [Ba]). Up to some error $O(h)$, the corresponding phase space localization is thus given by the support of its principal symbol $\theta(P)$:

$$\operatorname{supp}\,\theta(P) \;=\; \Big\{\mathrm{x}^* \in T^*\mathbb{R}^d;\ \exists\mu \in \operatorname{supp}\theta;\ \det\Big(P(\mathrm{x}^*) - \mu \mathrm{I}_m\Big) = 0\Big\}\,.$$

It is thus natural to consider the open set

$$E^*(\lambda,\epsilon_0) \;:=\; \bigcup_{\mu\in]\lambda-\epsilon_0;\lambda+\epsilon_0[} \Big\{\mathrm{x}^* \in T^*\mathbb{R}^d;\ \det\Big(P(\mathrm{x}^*) - \mu \mathrm{I}_m\Big) = 0\Big\}\,, \tag{2.2}$$

for some $\epsilon_0 > 0$, as an energy localization near $\lambda$. The energy shell of $P$ at energy $\lambda$ is

$$E^*(\lambda) \;:=\; \Big\{\mathrm{x}^* \in T^*\mathbb{R}^d;\ \det\Big(P(\mathrm{x}^*) - \lambda \mathrm{I}_m\Big) = 0\Big\}\,. \tag{2.3}$$

Now, let us explain what we mean by eigenvalues crossing. Here, we shall repeatedly make us of arguments from [K]. Let $x \in \mathbb{R}^d$. Using the min-max principle (cf. [RS4]), we find $m$ eigenvalues $\alpha_1(x) \le \cdots \le \alpha_m(x)$ of $M(x)$. We can extract from them $k(x)$ distinct eigenvalues of $M(x)$, with $k(x) \in \{1;\cdots;m\}$, namely $\lambda_1(x) < \cdots < \lambda_{k(x)}(x)$, and we denote by $\Pi_1(x),\cdots,\Pi_{k(x)}(x)$ the corresponding orthogonal eigenprojections. For $j \in \{1;\cdots;k(x)\}$, the number $m_j(x)$ of indices $\ell \in \{1;\cdots;m\}$ such that $\alpha_\ell(x) = \lambda_j(x)$ is the multiplicity of the eigenvalue $\lambda_j(x)$. Alternatively, we see that, in $\alpha_1(x) \le \cdots \le \alpha_m(x)$,

$m_j(x)$ is the number of equalities between the $(j-1)$th inequality and the $j$th one. Thus, we can define the functions $m_j$ for all $j \in \{1; \cdots ; m\}$. Notice that all $\alpha_\ell$, all $\lambda_j$ are continuous, and that (2.3) may be written as

$$E^*(\lambda) \;=\; \bigcup_{1\leq \ell\leq m} \Big\{ \mathrm{x}^* = (x, \xi) \in T^*\mathbb{R}^d;\; |\xi|^2 + \alpha_\ell(x) \;=\; \lambda \Big\}\,. \tag{2.4}$$

The eigenvalues crossing is defined as follows.

**Definition 2.1.** *We use the previous notation. The set of discontinuities of $k$ is also the union over $j$ of the set of discontinuities of $m_j$. By definition, this subset of $\mathbb{R}^d$ is called the eigenvalues crossing and is denoted by $\mathcal{C}$.*

Notice that, outside $\mathcal{C}$, the distinct eigenvalues and the corresponding orthogonal projections are smooth (cf. [K]). For the resolvent estimates at energy $\lambda$, the relevant part of the crossing is below the energy $\lambda$ (cf. [J1]). This leads to the

**Definition 2.2.** *We use the previous notation. Let $\mu \in \mathbb{R}$. By definition, the relevant eigenvalues crossing at energy $\mu$, denoted by $\mathcal{C}_r(\mu)$, is given by*

$$\begin{aligned} \mathcal{C}_r(\mu) \;:=\; \Big\{ x \in \mathbb{R}^d\,;\, \exists j \in \{1, \cdots, k(x)\}\,;\, \lambda_j(x) \leq \mu \\ \text{and } m_j \text{ is discontinuous at } x \Big\}\,. \end{aligned} \tag{2.5}$$

*The influence of the relevant eigenvalues crossing at energy $\mu$ takes place in the following subset $\mathcal{C}^*(\mu)$ of $E^*(\mu)$ (cf. (2.3)), called the crossing region at energy $\mu$,*

$$\begin{aligned} \mathcal{C}^*(\mu) \;:=\; \Big\{ \mathrm{x}^* = (x, \xi) \in \mathcal{C}_r(\mu) \times \mathbb{R}^d\,;\, \exists j \in \{1, \cdots, k(x)\}; \\ |\xi|^2 + \lambda_j(x) = \mu \text{ and } m_j \text{ is discontinuous at } x \Big\}\,. \end{aligned} \tag{2.6}$$

In Proposition 2.3 below, we give an useful characterization of the crossing region at some energy. To this end, we introduce

$$\chi \in C_0^\infty(\mathbb{R}; \mathbb{R}^+);\; \chi(0) \neq 0\,,\, \operatorname{supp}\chi = [-1; 1], \text{ and } \chi(t; \mu, \epsilon) := \chi\Big((t-\mu)/\epsilon\Big)\,, \tag{2.7}$$

for $\mu \in \mathbb{R}$, $\epsilon > 0$.

**Proposition 2.3.** *Let $\mu \in \mathbb{R}$, $\epsilon > 0$. Let $\chi$ and $\chi(\cdot; \cdot, \cdot)$ be as in (2.7). As a subset of $T^*\mathbb{R}^d$, let $S^*(\mu, \epsilon)$ be the support of the multiplication operator defined by*

$$C_0^\infty\Big(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C})\Big) \ni A \;\mapsto\; \Big(\mathrm{x}^* \;\mapsto\; \chi\Big(P(\mathrm{x}^*); \mu, \epsilon\Big)\, \Big[P(\mathrm{x}^*), A(\mathrm{x}^*)\Big]\, \chi\Big(P(\mathrm{x}^*); \mu, \epsilon\Big)\Big)\,.$$

*Then $S^*(\mu, \epsilon) \subset E^*(\mu, \epsilon)$ (cf. (2.2 )), $\epsilon \mapsto S^*(\mu, \epsilon)$ is non decreasing, and*

$$\mathcal{C}^*(\mu) \;=\; \bigcap_{\epsilon > 0} S^*(\mu, \epsilon)\,. \tag{2.8}$$

**Proof:** Let $\epsilon < \epsilon'$. Since $\chi(\cdot;\mu,\epsilon)/\chi(\cdot;\mu,\epsilon')$ is smooth, we see that the complement of $S^*(\mu,\epsilon)$ contains the complement of $S^*(\mu,\epsilon')$. This shows that $S^*(\mu,\epsilon) \subset S^*(\mu,\epsilon')$. Since $E^*(\mu,\epsilon) \supset \operatorname{supp}\chi(P(\cdot);\mu,\epsilon)$, $S^*(\mu,\epsilon) \subset E^*(\mu,\epsilon)$. It remains to prove (2.8). Denote by $\mathcal{D}^*(\mu)$ the right hand side of (2.8). From $S^*(\mu,\epsilon) \subset E^*(\mu,\epsilon)$ for all $\epsilon > 0$, we derive that $\mathcal{D}^*(\mu) \subset E^*(\mu)$. Recall that $\mathcal{C}^*(\mu) \subset E^*(\mu)$ (cf. (2.6) and (2.4)).
Let $\mathrm{x}_0^* = (x_0;\xi_0) \in E^*(\mu)\setminus\mathcal{C}^*(\mu)$. Near $x_0$, $M(x)$ has $N$ $(:= k(x_0))$ smooth distinct eigenvalues $\lambda_1(x) < \cdots < \lambda_N(x)$ such that $\lambda_N(x_0) \leq \mu$. The associated orthogonal projections $\Pi_1(x), \cdots, \Pi_N(x)$ are also smooth. For any $A \in C_0^\infty(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C}))$, which is supported close enough to $\mathrm{x}_0^*$, for small enough $\epsilon$, for all $\mathrm{x}^* \in T^*\mathbb{R}^d$,

$$\begin{aligned} & \chi\Big(P(\mathrm{x}^*);\mu,\epsilon\Big)\left[P(\mathrm{x}^*), A(\mathrm{x}^*)\right]\chi\Big(P(\mathrm{x}^*);\mu,\epsilon\Big) \\ = & \sum_{j,\ell\in\{1,\cdots,N\}} \chi\Big(|\xi|^2+\lambda_j(x);\mu,\epsilon\Big)\chi\Big(|\xi|^2+\lambda_\ell(x);\mu,\epsilon\Big)\Pi_j(x)\left[P(\mathrm{x}^*), A(\mathrm{x}^*)\right]\Pi_\ell(x) \\ = & \sum_{j\in\{1,\cdots,N\}} \chi^2\Big(|\xi|^2+\lambda_j(x);\mu,\epsilon\Big)\Pi_j(x)\left[|\xi|^2+\lambda_j(x), A(x,\xi)\right]\Pi_j(x) \;=\; 0\,. \end{aligned}$$

Thus $\mathrm{x}_0^* \notin \mathcal{D}^*(\mu)$.
Let $\mathrm{x}_0^* = (x_0;\xi_0) \in \mathcal{C}^*(\mu)$. Let $c > 0$ such that $\chi(t) \geq c$ if $|t| \leq 1/2$. Thanks to Definition 2.1, we can find, for any $\epsilon > 0$, some $y(\epsilon)$ in the ball centered at $x_0$ of radius $\epsilon$ such that, among the distinct eigenvalues $\lambda_1(y(\epsilon)), \cdots, \lambda_{k(y(\epsilon))}(y(\epsilon))$ of $M(x)$, there are two, say $\lambda_{j(\epsilon)}(y(\epsilon))$ and $\lambda_{\ell(\epsilon)}(y(\epsilon))$, such that, for $s \in \{j(\epsilon),\ell(\epsilon)\}$, $|\,|\xi_0|^2 + \lambda_s(y(\epsilon)) - \mu\,| < \epsilon/2$. Thus $\chi(|\xi|^2+\lambda_{j(\epsilon)}(y(\epsilon));\mu,\epsilon)\cdot\chi(|\xi|^2+\lambda_{\ell(\epsilon)}(y(\epsilon));\mu,\epsilon) \geq c^2$. Now, using a diagonalization of $M(x)$ at $y(\epsilon)$, we can construct $e$, the linear operator that exchanges an eigenvector of $\lambda_{j(\epsilon)}(y(\epsilon))$ with an eigenvector of $\lambda_{\ell(\epsilon)}(y(\epsilon))$. Let $\psi_\epsilon$ be a smooth, scalar cut-off function that localizes near $(y(\epsilon),\xi_0)$. If $A(x,\xi) = \psi_\epsilon(x,\xi)(\Pi_{j(\epsilon)}(x) + \Pi_{\ell(\epsilon)}(x))\cdot e$ $\cdot(\Pi_{j(\epsilon)}(x)+\Pi_{\ell(\epsilon)}(x))$, $\chi(P(\mathrm{x}^*);\mu,\epsilon)[P(\mathrm{x}^*),A(\mathrm{x}^*)]\chi(P(\mathrm{x}^*);\mu,\epsilon)$ is a non-zero function since it does not vanish at $y(\epsilon)$. Therefore $(y(\epsilon),\xi_0) \in S^*(\mu,\epsilon)$. Let $\epsilon_n \to 0$. For any $\epsilon > 0$, $(y(\epsilon_n),\xi_0) \in S^*(\mu,\epsilon_n) \subset S^*(\mu,\epsilon)$, for $n$ large enough, since $S^*(\mu,\cdot)$ is non-decreasing. Since $S^*(\mu,\epsilon)$ is closed, $(x_0,\xi_0) = \lim_n(y(\epsilon_n),\xi_0)$ belongs to $S^*(\mu,\epsilon)$. This shows that $(x_0,\xi_0) \in \mathcal{D}^*(\mu)$. $\square$

In view of Theorem 1.8, we need to explain in the general case what are the eigenvalues of the symbol $P$ and the corresponding Hamilton flows.

**Definition 2.4.** *We use the previous notation. For $j \in \{1,\cdots,m\}$, let $D_j := \{x \in \mathbb{R}^d;\ j \in \{1,\cdots,k(x)\}\}$. If $D_j \neq \emptyset$, the functions $\lambda_j, \Pi_j : D_j \longrightarrow \mathbb{R}$, $x \mapsto \lambda_j(x)$, $x \mapsto \Pi_j(x)$, are well defined. For such $j$, we define $p_j : D_j \times \mathbb{R}^d \longrightarrow \mathbb{R}$, $\mathrm{x}^* = (x,\xi) \mapsto |\xi|^2 + \lambda_j(x)$. If $\mu \in \mathbb{R}$, the relevant eigenvalues of $P$ at energy $\mu$ are the $p_j$ such that $p_j^{-1}(\mu) \neq \emptyset$.*
*Assume that $\mathcal{C}_r(\mu)$, the relevant eigenvalues crossing at energy $\mu$ (cf. (2.5)), is empty. Then each relevant eigenvalues $p_j$ of $P$ at energy $\mu$ is smooth on $p_j^{-1}(]\mu-\epsilon;\mu+\epsilon[)$, for some $\epsilon > 0$. For those $p_j$, the corresponding Hamilton flow can be defined as usual there.*

To close this subsection, we recall a well known fact on smooth projection-valued functions.

**Proposition 2.5.** *Let $\Omega$ be an open set of $\mathbb{R}^d$ and $\Pi : \Omega \longrightarrow \mathcal{M}_m(\mathbb{C})$ be a smooth, projection-valued function, i.e. $\Pi(x)^2 = \Pi(x)$, for all $x \in \Omega$. Then, for all $x \in \Omega$,*

$\Pi(x)(\nabla_x\Pi)(x)\Pi(x) = 0$. *Let* $N \in \mathbb{N}$ *and* $(\Pi_j)_{1\leq j\leq N}$ *be a family of smooth, projection-valued functions on* $\Omega$ *satisfying* $\Pi_j(x)\Pi_k(x) = \delta_{jk}\Pi_j(x)$*, for all* $x \in \Omega$ *and all* $(j,k) \in \{1;\cdots,N\}^2$*. Then,* $\sum_{j=1}^N \Pi_j(x)(\nabla_x\Pi_l)(x)\Pi_j(x) = 0$*, for all* $x \in \Omega$ *and all* $l \in \{1;\cdots,N\}$*.*

**Proof:** Expanding $0 = \Pi(\nabla_x(\Pi^2 - \Pi))\Pi$, we obtain the first property. Starting from $0 = \sum_j \Pi_j(\nabla_x(\Pi_l^2 - \Pi_l))\Pi_j = 2\Pi_l(\nabla_x\Pi_l)\Pi_l - \sum_j \Pi_j(\nabla_x\Pi_l)\Pi_j$ and applying the first property to $\Pi_l$, we obtain the second property. $\square$

## 2.2 Codimension 1 crossings.

Now we describe the model of codimension 1 crossings, the structure condition at infinity, and the invariance condition at the crossing. The precise definitions of these notions are given in Definitions 2.6, 2.8 and 2.9. To motivate the choice of this model, we want to explain how it follows from a reasonable "globalization" of the local definition of a codimension 1 crossing by [H].
For simplicity, we first restrict ourselves to crossings of two eigenvalues. We assume that there is a non-empty, closed submanifold $\mathcal{C}$ of $\mathbb{R}^d$ of codimension one such that, on each connected component of $\mathbb{R}^d\backslash\mathcal{C}$ the matrix $M(x)$ has a constant number of eigenvalues with constant multiplicity, so that we can label these eigenvalues with increasing order inside the connected component. We assume that the boundary of such a connected component (which is a part of $\mathcal{C}$) is the union of different crossings of exactly two eigenvalues. Let us explain this in detail. At each point $x_0 \in \mathcal{C}$, $\mathcal{C}$ is locally given by the equation $\tau = 0$, with $\tau \in C^\infty(\mathbb{R}^d;\mathbb{R})$ and $d\tau \neq 0$ on $\tau^{-1}(0)$. Near $x_0$ and on some connected component $\mathcal{U}_0$ of $\mathbb{R}^d \setminus \mathcal{C}$, there exist $j(1) < j(1)+1 < j(2) < j(2)+1 < \cdots < j(k) < j(k)+1$ such that, for all $i \in \{1,\cdots,k\}$, the limits on $\tau^{-1}(0)$ of the eigenvalues $\lambda_{j(i)}$ and $\lambda_{j(i)+1}$ (according to the previous labeling) coincide. Notice that the set $\mathcal{C}$ is the eigenvalues crossing in the sense of Definition 2.1. Inspired by [H], we require further the following matricial structure of $M(x)$ near $\mathcal{C}$.
Recall that $\mathrm{I}_m$ is the identity matrix in $\mathcal{M}_m(\mathbb{C})$. We denote by $\mathrm{tr}_m$ the normalized trace on $\mathcal{M}_m(\mathbb{C})$, i.e. $\mathrm{tr}_m\mathrm{I}_m = 1$. Let $x_0 \in \mathcal{C}$ and $i \in \{1,\cdots,k\}$. Near $x_0$, on $\mathcal{U}_0$, the sum $\mathcal{S}_i(x)$ of the spectral subspaces associated with $\lambda_{j(i)}(x)$ and $\lambda_{j(i)+1}(x)$ has constant dimension $m(i)$, up to $\overline{\mathcal{U}_0}$. We demand that, near $x_0$ and on $\overline{\mathcal{U}_0}$, the restriction $M_i(x)$ of $M(x)$ to $\mathcal{S}_i(x)$ is given by $(\mathrm{tr}_{m(i)}M_i(x))\mathrm{I}_{m(i)} + \tau(x)V_{j(i)}(x)$, where $V_{j(i)}$ is a smooth, self-adjoint matrix-valued function, defined on a vicinity of $x_0$ and having exactly two different eigenvalues there. Outside $\tau^{-1}(0)$, these eigenvalues are of course $((\lambda_{j(i)} - \mathrm{tr}_{m(i)}M_i)/\tau)(x)$ and $((\lambda_{j(i)+1}(x) - \mathrm{tr}_{m(i)}M_i)/\tau)(x)$. Using the smooth eigenvalues of $V_{j(i)}(x)$, we can smoothly extend $\lambda_{j(i)}(x)$ and $\lambda_{j(i)+1}(x)$ through $\mathcal{C}$ near $x_0$ such that they still are eigenvalues of $M(x)$. We recognize the codimension 1 crossings defined in [H].
By a connexity argument, we see that there exists some integer $N$ (the number of different eigenvalues) and $N$ globaly defined, smooth functions on $\mathbb{R}^d$, that we again denote by $\lambda_1,\cdots,\lambda_N$, such that, for all $x \in \mathbb{R}^d$ and all $j \in \{1,\cdots,N\}$, $\lambda_j(x)$ is an eigenvalue of $M(x)$, the multiplicity $m_j(x)$ of which is constant on $\mathbb{R}^d \setminus \mathcal{C}$ (see also [K], p. 108).
Outside $\mathcal{C}$, the orthogonal eigenprojection $\Pi_j(x)$ of $M(x)$ associated with $\lambda_j(x)$ is smooth

(cf. [K]). Let $j, k \in \{1, \cdots, N\}$ such that $j \neq k$ and $(\lambda_j - \lambda_k)^{-1}(0) \neq \emptyset$. Near any point $x_0 \in (\lambda_j - \lambda_k)^{-1}(0)$, the sum $\Pi_j(x) + \Pi_k(x)$ is smooth and has a constant range dimension $m(j,k) := m(j) + m(k)$, since only two different eigenvalues cross. According to the previous assumptions, there exist, near $x_0$, a smooth real, scalar function $\tau_{jk}$ and a smooth function $V_{jk}$, where $V_{jk}(x)$ is a self-adjoint matrix acting on $\mathcal{S}_{jk}(x)$, the range of $\Pi_j(x) + \Pi_k(x)$, with exactly two different eigenvalues, such that, for $x$ near $x_0$, the restriction $M_{jk}(x)$ of $M(x)$ to $\mathcal{S}_{jk}(x)$ has the form

$$M_{jk}(x) \; = \; \Big(\mathrm{tr}_{m(j,k)} M_{jk}(x)\Big) \mathrm{I}_{m(j,k)} \; + \; \tau_{jk}(x)\, V_{jk}(x)\,. \tag{2.9}$$

Near $x_0$, the set $(\lambda_j - \lambda_k)^{-1}(0)$ coincide with the zero set of $\tau_{jk}$, on which the differential of $\tau_{jk}$ does not vanish. Actually, we do not need that the differential of $\lambda_j - \lambda_k$ does not vanish on the crossing near $x_0$ (cf. Remark 2.11). Using the eigenprojections of $V_{jk}(x)$, we can smoothly extend $\Pi_j$ and $\Pi_k$ on $(\lambda_j - \lambda_k)^{-1}(0)$, near $x_0$. Therefore, we have $N$ globaly defined, smooth orthogonal projections $\Pi_1, \cdots, \Pi_N$. If the crossing is empty, there still exist, of course, globally smooth functions $\lambda_1, \cdots, \lambda_N$, $\Pi_1, \cdots, \Pi_N$, as above.
A careful analysis of the previous arguments shows that we can similarly deal with crossings of more than two eigenvalues and get a similar global situation, namely the situation described in

**Definition 2.6.** *We say that the potential $M$ is a model of codimension $1$ crossings if the following situation occurs. There exist some $N \in \{1; \cdots; m\}$, $N$ real-valued, smooth functions $\lambda_1, \cdots, \lambda_N$, and $N$ matrix-valued, smooth functions $\Pi_1, \cdots, \Pi_N$, such that, for all $x \in \mathbb{R}^d$, $\lambda_1(x), \cdots, \lambda_N(x)$ are the distinct eigenvalues of $M(x)$ and $\Pi_1(x), \cdots, \Pi_N(x)$ the associated orthogonal eigenprojections. The subset of $\mathbb{R}^d$*

$$\bigcup_{j,k \in \{1;\cdots;N\}, j \neq k} \Big(\lambda_j - \lambda_k\Big)^{-1}(0)$$

*is a codimension $1$ submanifold of $\mathbb{R}^d$ and is the eigenvalues crossing $\mathcal{C}$, in the sense of Definition 2.1. Furthermore, near each point $x_0 \in \mathcal{C}$, we require the following matricial structure. Let*

$$I \; = \; \Big\{ j \in \{1; \cdots; N\};\, \exists k \in \{1; \cdots; N\};\, k \neq j \;\; \text{and} \;\; x_0 \in (\lambda_j - \lambda_k)^{-1}(0) \Big\}\,,$$

*$\Pi(x) = \sum_{j \in I} \Pi_j(x)$, and $n \in \{1; \cdots; m\}$ be the dimension of the range of $\Pi(x)$. There is a scalar function $\tau \in C^\infty(\mathbb{R}^d; \mathbb{R})$ and a function $V \in C^\infty(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$, with values in the self-adjoint $n \times n$-matrices with empty eigenvalues crossing, such that, near $x_0$, $\mathcal{C}$ is given by $\tau^{-1}(0)$, $d\tau \neq 0$ on $\tau^{-1}(0)$, and the restriction of $M(x)$ to the range of $\Pi(x)$ is given by $\mathrm{tr}_n(M(x)\Pi(x))\mathrm{I}_n + \tau(x)V(x)$ (see (2.9)).*
*For $j \in \{1, \cdots, N\}$, the smooth Hamilton functions $p_j$ defined by $T^*\mathbb{R}^d \ni (x, \xi) \mapsto p_j(x, \xi) := |\xi|^2 + \lambda_j(x)$ are called the eigenvalues of $P$. For $j \in \{1, \cdots, N\}$ and $t \in \mathbb{R}$, we denote by $\phi_j^t$ the corresponding Hamilton flow at time $t$.*
*Finally, we say that the model satisfies the invariance condition at the crossing if, at any point on $\mathcal{C}$, all tangential derivatives of the orthogonal eigenprojections are zero. This means precisely the following, using the previous notation. For all $x_0 \in \mathcal{C}$, for all $j \in I$, for all $x \in \tau^{-1}(0)$, and for all $\xi \in \mathbb{R}^d$ with $\xi \cdot \nabla\tau(x) = 0$, $\xi \cdot \nabla\Pi_j(x) = 0$.*

**Remark 2.7.** *A model of codimension* 1 *crossings, for which $M$ is a radial function* (*i.e. that only depends on* $|x|$), *satisfies the invariance condition at the crossing. Indeed, the crossing $\mathcal{C}$ is an union of spheres centered at* 0 *with positive radius and near $x_0 \in \mathcal{C}$* (*with the notation of Definition 2.6*), *one can choose $\tau$ and $V$ radial. The eigenprojections are also radial. Therefore their tangential derivatives are zero.*

Now, we want to describe our requirement at infinity. Although it is perhaps easier to work (and to think) in the one point compactification of $\mathbb{R}^d$, we prefer to express our requirement in an elementary way.
By [K], we know that the eigenvalues $\lambda_j$ have a limit at infinity, which are eigenvalues of $M_\infty$. Two cases appear. We describe them in the following

**Definition 2.8.** *We use the previous notation. If there exists $c > 0$ and $R > 0$ such that, for all $j \neq k \in \{1, \cdots, N\}$ and for all $|x| > R$, $|\lambda_j(x) - \lambda_k(x)| \geq c$, we say that there is no "crossing at infinity". Otherwise, we say that there is a "crossing at infinity".*

In the second case, we require, in the spirit of Definition 2.6, a control on the matricial structure of $M$ near the "crossing at infinity" in the following way. Let

$$I_\infty = \Big\{ j \in \{1; \cdots; N\};\ \exists k \in \{1; \cdots; N\};\ k \neq j \ \text{ and } \lim_{|x| \to \infty} \lambda_j(x) - \lambda_k(x) = 0 \Big\}, \quad (2.10)$$

$\Pi(x) = \sum_{j \in I_\infty} \Pi_j(x)$, and $n \in \{1; \cdots; m\}$ be the dimension of the range of $\Pi(x)$. Notice that, at infinity, $\Pi(x)$ converges to some $\Pi_\infty$, an eigenprojection of $M_\infty$ (introduced in (2.1)). There exists some $R > 0$, a smooth, scalar function $\tau$, and a smooth function $V$, with values in the self-adjoint $n \times n$-matrices and with empty eigenvalues crossing, such that, on $\{y \in \mathbb{R}^d; |y| > R\}$, the restriction of $M(x)$ to the range of $\Pi(x)$ is given by $\mathrm{tr}_n(M(x)\Pi(x))\mathrm{I}_n + \tau(x)V(x)$. In view of (2.1), we require that there exists a self-adjoint $n \times n$-matrix $V_\infty$ such that, for all $\gamma \in \mathbb{N}^d$, for all $x \in \{y \in \mathbb{R}^d; |y| > R\}$,

$$\Big| \partial_x^\gamma \tau(x) \Big| + \Big\| \partial_x^\gamma \Big( V(x) - V_\infty \Big) \Big\|_n = O_\gamma\Big( \langle x \rangle^{-\rho - |\gamma|} \Big) . \quad (2.11)$$

This assumption roughly means that, at infinity, the function $\tau$ tends to zero while the matricial structure of $V$ keeps its properties up to the limit $V_\infty$. In other words, we treat the infinity of the configuration space $\mathbb{R}^d_x$ as a part of $\mathcal{C}$, at which we require (almost) the same assumptions as at finite distance on $\mathcal{C}$. Let us point out that (2.1) and (2.11) imply that, for all $j \in \{1; \cdots, N\}$, we can find an eigenvalue $\lambda_{j,\infty}$ of $M_\infty$ and an orthogonal projection $\Pi_{j,\infty}$, such that

$$\forall \gamma \in \mathbb{N}^d,\ \forall x \in \mathbb{R}^d,\ \Big| \partial_x^\gamma \Big( \lambda_j(x) - \lambda_{j,\infty} \Big) \Big| + \Big\| \partial_x^\gamma \Big( \Pi_j(x) - \Pi_{j,\infty} \Big) \Big\|_m = O_\gamma\Big( \langle x \rangle^{-\rho - |\gamma|} \Big). \quad (2.12)$$

Notice that the orthogonal eigenprojection of $M_\infty$ corresponding to $\lambda_{j,\infty}$ is the sum of the $\Pi_{k,\infty}$ over the set of the $k$ for which $\lambda_k \to \lambda_{j,\infty}$ at infinity. This leads to the following

**Definition 2.9.** *Consider a model $M$ of codimension* 1 *crossings, as defined in Definition 2.6. Assume it satisfies* (*2.12*) *and, if there is a "crossing at infinity"* (*in the sense of Definition 2.8*), *it also satisfies* (*2.11*), *then we say that the model $M$ satisfies the structure condition at infinity.*

**Remark 2.10.** *Let $M$ be a model of codimension 1 crossings such that there is "no crossing at infinity" (see Definition 2.8). Then (2.12) is a direct consequence of (2.1) (cf. [K]). Thus $M$ satisfies the structure condition at infinity.*

A simple example with $m = 2$, studied in [J4], is defined as follows. Let $\tau, u, v_1, v_2 \in C^\infty(\mathbb{R}^d; \mathbb{R})$ satisfying the estimate (2.11), such that $v_1^2 + v_2^2 > 0$ everywhere, and such that $\tau = 0$ defines a codimension 1 submanifold of $\mathbb{R}^d$. Set $M(x) = u(x)\mathrm{I}_2 + \tau(x)V(x)$ with

$$V(x) \ := \ \begin{pmatrix} v_1(x) & v_2(x) \\ v_2(x) & -v_1(x) \end{pmatrix} .$$

**Remark 2.11.** *In the spirit of Definitions 2.2 and 2.4, we may introduce an energy dependent version of Definitions 2.6, 2.8, and 2.9. Given an energy $\mu$, the requirements of each definition are imposed at points in $\mathcal{C}^*(\mu)$ and, at infinity, only on eigenvalues below $\mu$. This change does not affect the proofs of Theorems 1.2 and 1.3.*
*In Definitions 2.6 and 2.9, we may assume that the restriction of $M(x)$ to the range of $\Pi(x)$ is given by $\mathrm{tr}_n(M(x)\Pi(x))\mathrm{I}_n + f(\tau(x))V(x)$, where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is $C^\infty$ near 0 and vanishes at 0. This does not affect the proofs of Theorems 1.2 and 1.3 either.*

# 3 Semiclassical trapping.

The purpose of this section is to study the situation where the property ((1.3) and (1.4)) fails, for some $s > 1/2$, preparing that way a proof by contradiction of Theorem 1.2. In fact, we easily generalize to the general matricial case most of the results of our version of Burq's strategy (cf. [B]), developped in [J5]. As by-products, we derive Propositions 1.6 and 1.7. However an important feature resists to our analysis in the general case (cf. Subsection 3.2). For codimension 1 crossings, we need the invariance condition at the crossing (cf. Definition 2.6) to overcome the difficulty.

## 3.1 Generalization of scalar results.

In this subsection, we work in the general framework defined in Subsection 2.1 and generalize several results obtained in [J5]. They allow us to derive by contradition and in a similar way the proofs of Propositions 1.6 and 1.7.
Recall that $\|\cdot\|$ denotes the usual norm of $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$ and also the operator norm of the bounded operators on $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$. We consider the situation where the property ((1.3) and (1.4)) is false near $\lambda > \|M_\infty\|_m$, for some $s > 1/2$. This situation is interpreted as a "semiclassical trapping". Precisely, we assume the

**Hypothesis 1.** *There exist a sequence $(h_n)_n \in ]0; h_0]^{\mathbb{N}}$ tending to zero, a sequence $(f_n)_n$ of nonzero, $\mathbb{C}^m$-valued functions of the domain of $\Delta_x \mathrm{I}_m$, and a sequence $(z_n)_n \in \mathbb{C}^{\mathbb{N}}$ with $\Re(z_n) \to \lambda > \|M_\infty\|_m \geq 0$ and $0 \leq \Im(z_n)/h_n \to r_0 \geq 0$, such that*

$$\|\langle x\rangle^{-s} f_n\| \ = \ 1 \quad \text{and} \quad \Big\|\langle x\rangle^{s}\Big(\hat{P}(h_n) - z_n\Big)f_n\Big\| \ = \ o(h_n) .$$

*Furthermore the* $\mathrm{L}^2$*-bounded sequence* $(\langle x\rangle^{-s} f_n)_n$ *is pure. We denote by* $\mu_s$ *its semiclassical measure and we set* $\mu := \langle x\rangle^{2s}\mu_s$.

**Remark 3.1.** *If, for some* $s > 1/2$, *(1.3) would be false at some energy* $\lambda_n \to \lambda$ *for some sequence* $h_n \to 0$, *then, by Mourre's theory (cf. [M, CFKS]),* $\lambda_n$ *would belong to the pure point spectrum of* $\hat{P}(h_n)$, *for any* $n$. *Then, there would exist, for each* $n$, *an eigenvalue* $\lambda'_n$ *of* $\hat{P}(h_n)$ *such that* $\lambda'_n \to \lambda$. *Choosing, for each* $n$, *a corresponding eigenvector with a suitable normalization as* $f_n$ *and setting* $z_n = \lambda'_n$, *Hypothesis 1 would also hold true in this case. The generalization to matricial Schrödinger operators of the result by Froese and Herbst (cf. [FH, CFKS]) tells us that this case does not occur. This does not affect the proof of Theorem 1.2 since it is based on Hypothesis 1.*

We shall use well-known tools of semiclassical analysis, like $h$-pseudodifferential operators and semiclassical measure. We refer to [DG, GL, N, R] for details. For matrix-valued symbolic calculus, we refer to [Ba, J2]. The notation and the most important facts we need are recalled below.
For $(r,t) \in \mathbb{R}^2$, we consider the class of symbols $\Sigma_{r,t}$, composed of the smooth functions $A : T^*\mathbb{R}^d \longrightarrow \mathcal{M}_m(\mathbb{C})$ such that

$$\forall \gamma = (\gamma_x, \gamma_\xi) \in \mathbb{N}^{2d}\,,\ \exists C_\gamma > 0\,;\ \sup_{(x,\xi)\in T^*\mathbb{R}^d} \langle x\rangle^{-r+|\gamma_x|}\,\langle \xi\rangle^{-t+|\gamma_\xi|}\Big\|(\partial^\gamma A)(x,\xi)\Big\|_m \ \le\ C_\gamma\,. \quad (3.1)$$

For such symbol $A$, we can define its Weyl $h$-quantization, denoted by $A_h^w$, which acts on $u \in C_0^\infty(\mathbb{R}^d;\mathbb{C}^m)$ as follows.

$$\Big(A_h^w u\Big)(x) \ =\ (2\pi h)^{-n}\int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)/h} A\Big((x+y)/2, \xi\Big)\cdot u(y)\, dy d\xi\,. \quad (3.2)$$

By Calderon-Vaillancourt theorem, it extends to a bounded operator on $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^m)$, uniformly w.r.t. $h$, if $r,t \le 0$ (cf. [Ba, J2]). If $\theta \in C_0^\infty(\mathbb{R};\mathbb{R})$, one can show (cf. [Ba]) that

$$\theta\Big(\hat{P}(h)\Big) \ =\ \Big(\theta(P)\Big)_h^w \ +\ h\, Q_h^w \ +\ h^2\, R(h) \quad (3.3)$$

where $Q \in \Sigma_{-1,0}$ and where, for all $k \in \mathbb{R}$, $\langle x\rangle^{k+2}R(h)\langle x\rangle^{-k}$ is bounded operator on $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^m)$, uniformly w.r.t $h$. We point out that we can use Helffer-Sjöstrand's formula (cf. [DG]) to express $\theta(P)$ and to show that $\nabla_{x,\xi}(\theta(P))$ is supported in $\mathrm{supp}\,\theta'(P)$.
The measures $\mu_s, \mu$ can be viewed as nonnegative distributions $\mu_s, \mu \in \mathcal{D}'(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C}))$ of order 0, that satisfy, for all $A \in C_0^\infty(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C}))$,

$$\lim_{n\to\infty}\Big\langle \langle x\rangle^{-s} f_n\,,\, A_{h_n}^w\langle x\rangle^{-s} f_n\Big\rangle \ =\ \mu_s(A) \quad \text{and} \quad \lim_{n\to\infty}\Big\langle f_n\,,\, A_{h_n}^w f_n\Big\rangle \ =\ \mu(A)\,. \quad (3.4)$$

**Proposition 3.2.** *Under the previous conditions, the support of* $\mu$, *denoted by* $\mathrm{supp}\,\mu$, *satisfies* $\mathrm{supp}\,\mu \subset E^*(\lambda)$ (cf. *(2.3)*) *and* $\|f_n\|^2\Im(z_n)/h_n \to 0$. *In particular,* $r_0 = 0$.

**Proof:** It suffices to follow the arguments in [J5] and to use (3.3). □

**Remark 3.3.** *As noticed by V. Bruneau, the negation of (1.4) and the fact that* $\|R(z_n)\| \leq 1/|\Im(z_n)|$ *imply that* $r_0 = 0$*. However, we need the stronger fact* $\|f_n\|^2 \Im(z_n)/h_n \to 0$ *for the proof of Proposition 3.4 (see [J5] ).*

**Proposition 3.4.** *Let* $\rho' \in ]0; \min(1; \rho)[$*, (cf. (2.1)). There exists* $R_0 > 0$ *such that*

$$\lim_{n\to\infty} \int_{|x|>R_0} \langle x\rangle^{-(1+\rho')} \, |f_n(x)|^2 \, dx \;=\; 0\,.$$

*In particular, the measure* $\mu$ *is nonzero and has a compact support satisfying* $\operatorname{supp}\mu \subset E^*(\lambda) \cap \{\mathrm{x}^* = (x, \xi) \in T^*\mathbb{R}^d; |x| \leq R_0\}$ (cf. (*2.3*)).

**Proof:** Again the arguments in [J5] apply. To see this, we use the symbolic calculus of $\Sigma_{r,t}$ and the following important fact. For any **scalar** $a \in \Sigma_{r,t}$, $ih^{-1}[\hat{P}(h), (a\mathrm{I}_m)_h^w]$ is of order 0 in $h$ since $[P, a\mathrm{I}_m] = 0$ everywhere and its principal symbol w.r.t. $h$ is given by $(2\xi \cdot \nabla_x a)\mathrm{I}_m \;-\; \nabla_\xi a \cdot \nabla_x M$. □

**Proof of Proposition 1.6:** We assume that the property ((1.3) and (1.4)) holds true for some $s' > 1/2$ and that it fails for some $s > 1/2$. Thus, we have a "semiclassical trapping" for $s$ and we use the notation of Hypothesis 1 (at the beginning of Subsection 3.1). Let $\chi \in C_0^\infty(\mathbb{R}^d; \mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi^{-1}(0) = \{x; |x| \geq 2\}$, and $\chi^{-1}(1) = \{x; |x| \leq 1\}$. For $R > R_0$, we set $\chi_R(x) = \chi(x/R)$. By Proposition 3.4 and Hypothesis 1,

$$\liminf_{n\to\infty} \left\| \langle x\rangle^{-s'} \chi_R f_n \right\|^2 \;>\; 0\,.$$

Furthermore,

$$\langle x\rangle^{s'} \Big(\hat{P}(h_n) - z_n\Big)\chi_R f_n \;=\; \langle x\rangle^{s'} \chi_R \Big(\hat{P}(h_n) - z_n\Big) f_n \;+\; h_n \langle x\rangle^{s'} h_n^{-1} [\hat{P}(h_n), \chi_R]\, f_n\,. \quad (3.5)$$

Since $\chi_R$ is a scalar function and varies in $\{x; 2R > |x| > R_0\}$, we see, using Proposition 3.4 and Proposition 3.2, that the $\mathrm{L}^2$-norm of the last term in (3.5) is $o(h_n)$. By Hypothesis 1, so is the second term. Therefore, Hypothesis 1 is satisfied for $s'$. This contradicts the property ((1.3) and (1.4)) for $s'$. □

**Proof of Proposition 1.7:** It follows from Mourre theory (cf. [M, CFKS]) that, for all $r > 1/2$, the boundary values $\langle x\rangle^{-r} R(\lambda' - i0) \langle x\rangle^{-r}$ exist for $\lambda'$ close enough to $\lambda$ and for $h$ small enough. We prove that the estimate (1.4) for $R(\lambda + i0)$ and $s > 1/2$ implies the same estimate for $R(\lambda - i0)$ and the same $s$. Looking again for a contradiction, we assume that (1.4) holds true for $R(\lambda + i0)$ and fails for $R(\lambda - i0)$. Thus Hypothesis 1 (at the beginning of Subsection 3.1) holds true if we replace the condition $\Im(z_n) \geq 0$ by $\Im(z_n) \leq 0$. Of course, the previous results (Propositions 3.2 and 3.4) are still true. We first remark that $\|f_n\| \Im(z_n)/h_n \to 0$ by Propositions 3.2 since

$$\|f_n\| \cdot \frac{|\Im(z_n)|}{h_n} \;\leq\; \max\Big(\|f_n\|, 1\Big) \cdot \frac{|\Im(z_n)|}{h_n} \leq\; \max\Big(\|f_n\|^2, 1\Big) \cdot \frac{|\Im(z_n)|}{h_n}\,.$$

Let $R > R_0$ and $\chi_R$ be as in the proof of Proposition 1.6. By Proposition 3.4, $\|\langle x\rangle^{-s} f_n\| \geq \|\langle x\rangle^{-s} \chi_R f_n\| = \|\langle x\rangle^{-s} f_n\|(1 + o(1))$. In particular, $\|\langle x\rangle^{-s} \chi_R f_n\| \to 1$. Furthermore,

$$\begin{aligned}\langle x\rangle^{s} \Big(\hat{P}(h_n) - \overline{z_n}\Big)\chi_R f_n \;&=\; \langle x\rangle^{s} \chi_R \Big(\hat{P}(h_n) - z_n\Big) f_n \;+\; h_n \langle x\rangle^{s} h_n^{-1} [\hat{P}(h_n), \chi_R]\, f_n \\ &\quad + 2ih_n \cdot \Big(\Im(z_n)/h_n\Big) \cdot \langle x\rangle^{s} \chi_R f_n\,. \quad (3.6)\end{aligned}$$

As in the proof of Proposition 1.6, the $\mathrm{L}^2$-norms of the two first terms on the r.h.s of (3.6) are $o(h_n)$. Since $\chi_R$ is compactly supported and $\|f_n\|\Im(z_n)/h_n \to 0$, we see that the $\mathrm{L}^2$-norm of the last one is also $o(h_n)$. Since $\Im(\overline{z_n}) \geq 0$, Hypothesis 1 holds true. This contradicts the property ((1.3) and (1.4)) for $R(\lambda + i0)$. $\square$

## 3.2 A matricial difficulty.

Trying to follow the other arguments in [J5], we meet, for the general case, a serious difficulty which is produced by the matricial nature of the symbol $P$.
As in [J5], we want to exploit properties of the commutator $ih^{-1}[\hat{P}(h), A_h^w]$, for symbol $A \in C_0^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$. To this end, we use the symbolic calculus for $\Sigma_{r,t}$ (cf. (3.1)). If one keeps the order of the symbols, the composition formula for the corresponding Weyl $h$-pseudodifferential operators is the same as in the scalar case (see [Ba, J2, DG, R]). In particular, one can show (see [Ba, J2]) that, for $A \in \Sigma_{0,0}$,

$$ih^{-1}\Big[\hat{P}(h)\,,\,A_h^w\Big] \;=\; h^{-1}\Big(i[P,A]\Big)_h^w \;-\; \Big(\{P,A\}\Big)_h^w \;+\; h\,R(A;h)\,, \tag{3.7}$$

where, for all $k \in \mathbb{R}$, $\langle x\rangle^k R(A;h)\langle x\rangle^{-k}$ is a bounded operator on $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^m)$, uniformly w.r.t $h$, and where

$$\begin{aligned}\{P,A\} \;&:=\; (1/2)\Big(\nabla_\xi P\cdot\nabla_x A - \nabla_x P\cdot\nabla_\xi A\Big) \;+\; (1/2)\Big(\nabla_x A\cdot\nabla_\xi P - \nabla_\xi A\cdot\nabla_x P\Big)\\ &=\; 2\xi\cdot\nabla_x A \;-\; (1/2)\Big(\nabla_x M\cdot\nabla_\xi A + \nabla_\xi A\cdot\nabla_x M\Big)\,.\end{aligned} \tag{3.8}$$

Here we recognize a symmetrized, matricial version of the usual Poisson bracket. In contrast to the scalar case, the second term in (3.7) is nonzero in general and is responsible for the difficulty we mentionned. Instead of proving $\{P,\mu\} = 0$ (in the distributional sense) as in the scalar case (cf. [J5]), we only have the

**Proposition 3.5.** *Consider the general case (see Subsection 2.1) under Hypothesis 1 (see Subsection 3.1). For any $A \in C_0^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$, there exists*

$$\nu(A) \;:=\; \lim_{n\to\infty}\Big\langle f_n\,,\,h_n^{-1}\Big(i[P,A]\Big)_{h_n}^w f_n\Big\rangle\,. \tag{3.9}$$

*$\nu \in \mathcal{D}'(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$ and satisfies, in the distributional sense, $-\{P,\mu\} = \nu$. This means that, for any $A \in C_0^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$, $\nu(A) = \mu(\{P,A\})$, where $\{P,A\}$ is defined in (3.8). Furthermore, $\operatorname{supp}\nu \subset \mathcal{C}^*(\lambda)\cap\operatorname{supp}\mu$ ($\mathcal{C}^*(\lambda)$ is defined in Definition 2.2) and $\nu(A) = 0$ if $A$ commutes with $P$ near $\operatorname{supp}\nu$.*

**Proof:** Let $A \in C_0^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$. If we expand the commutator in the scalar product $\langle f_n, ih_n^{-1}[\hat{P}(h_n), A_{h_n}^w]f_n\rangle$, then we see that this quantity tends to 0, thanks to Hypothesis 1, since $\langle x\rangle^s A_{h_n}^w\langle x\rangle^{-s}$ and $\langle x\rangle^s A_{h_n}^w\langle x\rangle^s$ are bounded operators, uniformly w.r.t. $n$. Using (3.7) and (3.4), we deduce from this that the limit (3.9) exists and is $\mu(\{P,A\})$, since $\langle x\rangle^s R(A;h_n)\,\langle x\rangle^{-s}$ is also a bounded operator, uniformly w.r.t. $n$. We have proved that

$-\{P,\mu\} = \nu$ and, in particular, that $\operatorname{supp}\nu \subset \operatorname{supp}\mu$.
Let $A \in C_0^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$ that commutes with $P$ near $\mathcal{C}^*(\lambda)$. Let $\chi$, $\chi(\cdot;\cdot,\cdot)$ be as in (2.7) such that $\chi = 1$ near 0. By Proposition 2.3, there exists some $\epsilon > 0$ such that

$$T^*\mathbb{R}^d \ni \mathrm{x}^* \ \mapsto\ \chi\Big(P(\mathrm{x}^*);\lambda,\epsilon\Big)\,\Big[P(\mathrm{x}^*), A(\mathrm{x}^*)\Big]\,\chi\Big(P(\mathrm{x}^*);\lambda,\epsilon\Big)$$

is identically zero. Using the energy localization of the $f_n$, (3.3), and (3.4),

$$\begin{aligned}
&\langle f_n\,,\, h_n^{-1}\Big(i[P,A]\Big)^w_{h_n} f_n\rangle \\
&= \ \Big\langle f_n\,,\, h_n^{-1}\chi\Big(\hat{P}(h_n);\lambda,\epsilon\Big)\,\Big(i[P,A]\Big)^w_{h_n}\chi\Big(\hat{P}(h_n);\lambda,\epsilon\Big)\,f_n\Big\rangle \ + \ o(1) \\
&= \ 0\cdot h_n^{-1} \ + \ \mu\bigg(\chi\Big(P;\lambda,\epsilon\Big)\,i[P-\lambda,A]\,Q \ + \ Q_1 \ + \ Q\,i[P-\lambda,A]\chi\Big(P;\lambda,\epsilon\Big)\bigg) \ + \ o(1)\,,
\end{aligned}$$

where $Q_1$ is supported in $\operatorname{supp}\chi'(P;\lambda,\epsilon)$. Thanks to Proposition 3.2, we obtain $\nu(A) = 0$. Now, if $A \in C_0^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$ is supported away from $\mathcal{C}^*(\lambda)$, then it commutes with $P$ near $\mathcal{C}^*(\lambda)$. Thus $\nu(A) = 0$ and $\operatorname{supp}\nu \subset \mathcal{C}^*(\lambda)$.
Let $A \in C_0^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$ that commutes with $P$ near $\operatorname{supp}\nu$. Then we can find $\psi \in C^\infty(T^*\mathbb{R}^d; \mathbb{R})$ such that $\psi = 1$ near $\operatorname{supp}\nu$ and $[P,\psi A] = 0$ everywhere. Thus $\nu(A) = \nu(\psi A) = 0$, by (3.9). □
A microlocalized version of the equation $-\{P,\mu\} = \nu$ is given in

**Proposition 3.6.** *Consider the general case (see Subsection 2.1) under Hypothesis 1 (see Subsection 3.1). Let $\mathrm{x}_0^* = (x_0,\xi_0) \in \mathcal{C}^*(\lambda)$ and, using the notation in Definition 2.2, let*

$$I \ := \ \Big\{j \in \{1,\cdots,k(x_0)\};\quad |\xi_0|^2 + \lambda_j(x_0) = \lambda \text{ and } m_j \text{ is discontinuous at } x_0\Big\}\,.$$

*Close enough to $\mathrm{x}_0^*$, the orthogonal projection onto the sum over $j \in I$ of the spectral subspaces associated with $\lambda_j(x)$, denoted by $\Pi_I(x)$, is smooth and we set $A_I = \Pi_I A \Pi_I$, for $A \in C^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$. For $A \in C_0^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$, supported close enough to $\mathrm{x}_0^*$,*

$$\nu(A_I) \ = \ \mu\Big(\{P_I, A_I\}_I\Big)\,. \tag{3.10}$$

**Proof:** Since the eigenvalues $\lambda_k$ are continuous, the $\lambda_j$ for $j \in I$ are separated from the rest of the spectrum of $M$ in some vicinity of $x_0$. Therefore $\Pi_I$ is smooth there.
Let $A \in C_0^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$ with support close enough to $\mathrm{x}_0^*$. We follow the beginning of the proof of Proposition 3.5, applied to $\langle f_n, ih_n^{-1}[\hat{P}(h_n), (A_I)^w_{h_n}]f_n\rangle$. To get (3.10), it suffices to show that

$$\Big\langle f_n\,,\,\Big(\{P,A_I\}\Big)^w_{h_n} f_n\Big\rangle \ = \ \Big\langle f_n\,,\,\Big(\{P_I,A_I\}_I\Big)^w_{h_n} f_n\Big\rangle \ + \ o(1)\,. \tag{3.11}$$

Let $\chi$, $\chi(\cdot;\cdot,\cdot)$ be as in (2.7) such that $\chi = 1$ near 0. Taking $\epsilon > 0$, we set $\chi_0 = \chi(\cdot;\lambda,\epsilon)$. Using the energy localization of the $f_n$ and (3.3), we obtain

$$\begin{aligned}
\Big\langle f_n\,,\,\Big(\{P,A_I\}\Big)^w_{h_n} f_n\Big\rangle \ &:= \ \Big\langle f_n\,,\,\Big(\chi_0(P)\{P,A_I\}\chi_0(P)\Big)^w_{h_n} f_n\Big\rangle \ + \ o(1) \\
&= \ \Big\langle f_n\,,\,\Big(\chi_0(P_I)\{P,A_I\}\chi_0(P_I)\Big)^w_{h_n} f_n\Big\rangle \ + \ o(1) \\
&= \ \Big\langle f_n\,,\,\Big(\chi_0(P_I)\{P,A_I\}_I\chi_0(P_I)\Big)^w_{h_n} f_n\Big\rangle \ + \ o(1)\,,
\end{aligned}$$

since $A$ vanishes on the support of $\chi_0(P(1-\Pi_I))$. Using the arguments backwards,

$$\Big\langle f_n\,,\,\Big(\{P,A_I\}\Big)^w_{h_n} f_n\Big\rangle \;=\; \Big\langle f_n\,,\,\Big(\{P,A_I\}_I\Big)^w_{h_n} f_n\Big\rangle \;+\; o(1)\,.$$

Thanks to Proposition 2.5, $\Pi_I\{P(1-\Pi_I),A_I\}\Pi_I = 0$, yielding (3.11). $\square$

## 3.3 Codimension 1 crossings with an invariance condition.

Now we focus on codimension 1 crossings. This allows us to consider smooth symbols that commute with $P$ near the crossing and thus to remove the influence of the distribution $\nu$ of Proposition 3.5. However, we do need the invariance condition at the crossing to perform the end of the proof of Theorem 1.2.
In the framework defined in Subsection 2.2, we first want to derive other properties of $\mu$ near some point $\mathrm{x}_0^* = (x_0,\xi_0) \in \mathcal{C}^*(\lambda)$ ($\mathcal{C}^*(\lambda)$ being defined in Definition 2.2).

**Proposition 3.7.** *Consider the model of codimension* 1 *crossing* (*see Definition 2.6*) *under Hypothesis 1* (*see Subsection 3.1*)*. Let* $\mathrm{x}_0^* = (x_0,\xi_0) \in \mathcal{C}^*(\lambda)$*. We use the notation of Proposition 3.6 and of Definition 2.6. We denote by* $\mathbb{1}_{\mathcal{C}^*(\lambda)}$ *the characteristic function of* $\mathcal{C}^*(\lambda)$*. For all* $A \in C_0^\infty(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C}))$*, supported close enough to* $\mathrm{x}_0^*$*,* $\mathbb{1}_{\mathcal{C}^*(\lambda)}\mu((\xi\cdot\nabla\tau)A) = 0$*. In particular, near* $\mathrm{x}_0^*$*,*

$$\operatorname{supp}\mathbb{1}_{\mathcal{C}^*(\lambda)}\mu \;\subset\; \Big\{\mathrm{x}^* = (x,\xi) \in \mathcal{C}^*(\lambda)\,;\,\xi\cdot\nabla\tau(x) \;=\; 0\Big\}\,. \tag{3.12}$$

*Furthermore, if the model of codimension* 1 *crossing satisfies the invariance condition at the crossing* (*see Definition 2.6*) *and if* $A = \sum_{j\in I} a_j\Pi_j$*, for smooth, scalar functions* $a_j$*, localized near* $\mathrm{x}_0^*$*, then*

$$\mu\left(\sum_{j\in I}\{p_j,a_j\}\Pi_j\right) \;=\; 0\,. \tag{3.13}$$

**Proof:** For $A \in C^\infty(T^*\mathbb{R}^d;\mathcal{M}_m(\mathbb{C}))$, we set $\pi_I(A) = \sum_{j\in I}\Pi_j A\Pi_j$. Let $A$ be localized near $\mathrm{x}_0^*$. Since $[\pi_I(A),P] = 0$ and $\Pi_I\pi_I(A)\Pi_I = \pi_I(A)$, Proposition 3.5 and (3.10) yield

$$\mu\Big(\{P_I,\pi_I(A)\}_I\Big) \;=\; 0\,. \tag{3.14}$$

This is still true if we replace $A$ by $\tau\chi(\tau/\epsilon)A$, where $\chi \in C_0^\infty(\mathbb{R};\mathbb{R})$ with $\chi = 1$ near 0 and $\epsilon > 0$. By the dominated convergence theorem,

$$\lim_{\epsilon\to 0^+}\mu\Big((\tau/\epsilon)\chi'(\tau/\epsilon)(2\xi\cdot\nabla\tau)\pi_I(A)\Big) \;=\; 0\,.$$

Thus $\mathbb{1}_{\tau=0}\mu((2\xi\cdot\nabla\tau)\pi_I(A)) = 0$, where $\mathbb{1}_{\tau=0}$ is the characteristic function of $\{(x,\xi) \in T^*\mathbb{R}^d;\tau(x) = 0\}$, which coincides locally with $\mathcal{C}\times\mathbb{R}^d$, the crossing viewed in $T^*\mathbb{R}^d$ (cf. Definition 2.6). This implies that $\mathbb{1}_{\mathcal{C}^*(\lambda)}\mu((2\xi\cdot\nabla\tau)\pi_I(A)) = 0$. Thus, near $\mathrm{x}_0^*$, $\mathbb{1}_{\mathcal{C}^*(\lambda)}\pi_I(\mu)$ is supported in the r.h.s of (3.12).

But, near $x_0^*$, $\mathrm{supp}\,\mu$ is contained in $\mathrm{supp}\,\pi_I(\mu)$. Indeed, let $B$ be non-negative and localized near $x_0^*$. We have $0 \leq \mu(B) = \mu(B_I)$, by energy localization of $\mu$ (cf. Proposition 3.2). Let $A = \langle B\rangle \Pi_I$. Since $A = \pi_I(A) \geq B_I$, we obtain $0 \leq \mu(B) \leq \mu(\pi_I(A))$. Since $\mathrm{supp}\,A \subset \mathrm{supp}\,B$, this yields $\mathrm{supp}\,\mu \subset \mathrm{supp}\,\pi_I(\mu)$. This implies (3.12).
Now let $A := \sum_{j\in I} a_j \Pi_j = A_I = \pi_I(A)$ be localized near $x_0^*$. By Proposition 2.5,

$$\Pi_I \{P_I, A\} \Pi_I \;=\; \sum_{j\in I} \{p_j, a_j\} \Pi_j \;+\; \sum_{j,k\in I, j\neq k} \Pi_j \Big(Q_1 \;+\; \tau Q_2\Big)\Pi_k \,, \tag{3.15}$$

where $Q_1 = \sum_{l\in I} a_l(2\xi\cdot\nabla\Pi_l)$ and $Q_2$ is smooth. Away from $\mathcal{C}^*(\lambda)$, the last term of (3.15)

$$= \sum_{j,k\in I, j\neq k} (p_j - p_k)^{-1}\Pi_j[P\,,\, Q_1 \;+\; \tau Q_2]\Pi_k \;=\; [P-\lambda, Q_3]\,,$$

for some smooth $Q_3$. It is thus annihilated by $\mu$ (cf. Proposition 3.2). Therefore,

$$\mu\Bigg(\sum_{j,k\in I, j\neq k} \Pi_j\Big(Q_1 \;+\; \tau Q_2\Big)\Pi_k\Bigg) \;=\; \mathbb{1}_{\mathcal{C}^*(\lambda)}\mu\Bigg(\sum_{j,k\in I, j\neq k} \Pi_j\, Q_1\, \Pi_k\Bigg)\,.$$

But, by the invariance condition (cf. Definition 2.6) and (3.12), $Q_1$ precisely vanishes on the support of $\mathbb{1}_{\mathcal{C}^*(\lambda)}\mu$. Now (3.14) and (3.15) imply (3.13). □

**Proof of Theorem 1.2:** We want to prove the property ((1.3) and (1.4)) by contradiction. Thus, we assume that Hypothesis 1 (at the beginning of Subsection 3.1) holds true and can apply the results above. In particular, we know that the measure $\mu$ is compactly supported in the energy shell $E^*(\lambda)$ (cf. (2.3)) and is nonzero (cf. Proposition 3.4). As in [J5], we are going to show that the non-trapping condition on the flows of the eigenvalues actually implies that $\mu = 0$, yielding the desired contradiction.
By this non-trapping condition, we can find $c > 0$ and, for all $j \in \{1, \cdots, N\}$, a smooth scalar function $a_j$ such that $\{p_j, a_j\} \geq c$ on $p_j^{-1}(\lambda) \cap \mathrm{supp}\,\mu$ (cf. [GM, J5]). If

$$\mu\Bigg(\sum_{j=1}^{N} \{p_j, a_j\}\, \Pi_j\Bigg) \;=\; 0\,, \tag{3.16}$$

then $0 \geq c\mu(I_m)$, yielding $\mu = 0$. Therefore, it suffices to show (3.16).
Using a partition of unity, it suffices to show (3.16) for functions $a_j$ localized away from $\mathcal{C}^*(\lambda)$, the crossing region at energy $\lambda$ defined in (2.6), and for functions $a_j$ localized near any $x_0^* \in \mathcal{C}^*(\lambda)$. In the second case, the localization implies that

$$\mu\Bigg(\sum_{j=1}^{N} \{p_j, a_j\}\, \Pi_j\Bigg) \;=\; \mu\Bigg(\sum_{j\in I} \{p_j, a_j\}\, \Pi_j\Bigg)$$

which equals to zero, by Proposition 3.7. In the first case, the arguments in the proofs of Propositions 3.5 and 3.7 directly show (see also [J4]) that

$$\mu\Bigg(\sum_{j=1}^{N} \{p_j, a_j\}\, \Pi_j\Bigg) \;=\; \mu\Bigg(\Big\{P\,,\, \sum_{j\in I} a_j\Pi_j\Big\}\Bigg) \;=\; 0\,. \qquad \square$$

# 4 Toward the non-trapping condition.

Now we come to the proof of Theorem 1.3. Starting from the property ((1.3) and (1.4)) at energy $\lambda > \|M_\infty\|_m$, we first apply Proposition 1.7 to have the same property applied to $R(\lambda - i\epsilon)$ instead of $R(\lambda + i\epsilon)$. Then we want to derive the non-trapping condition. To this end, we follow the strategy of [W] based on Egorov's theorem and on the use of coherent states. For empty crossing, a minor change in Wang's arguments give the non-trapping condition, since we can derive a weak form of Egorov's theorem (see Subsection 4.1). In the general case or even for our model of codimension 1 crossing (cf. Definition 2.6), we are not able to follow the same lines. However, for our codimension 1 crossing with structure condition at infinity (cf. Definitions 2.6 and 2.9), we can adapt the previous strategy and extract the non-trapping condition (see Subsection 4.2).

## 4.1 Wang's strategy.

First, we recall Wang's strategy and then show that, if the relevant eigenvalues crossing at energy $\lambda$ (cf. Definition 2.2) is empty, it gives the desired non-trapping condition (cf. Proposition 4.1). This completes the proof of Theorem 1.8. The effect of the crossing will be considered in Subsection 4.2.
Consider $s > 1/2$, $\lambda \in \mathbb{R}$, $h_0 > 0$, and an interval $I$ about $\lambda$, such that the boundary values $\langle x\rangle^{-s} R(\mu + i0)\langle x\rangle^{-s}$ and $\langle x\rangle^{-s} R(\mu - i0)\langle x\rangle^{-s}$ exist on $I$, for $h \in ]0; h_0]$ (cf. (1.3)). We assume further that, for $\mu$ in the same interval $I$ and $h \in ]0; h_0]$, the resolvent estimate (1.4) and the same estimate for $R(\mu - i0)$ hold true. Recall that $\|\cdot\|$ denotes the operator norm of the bounded operators on $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$. Following [W], we interpret the resolvent estimates by means of Kato's notion of locally $\hat{P}(h)$-smoothness (see [RS4]). These estimates imply that $\langle x\rangle^{-s}$, for $s > 1/2$, is $\hat{P}(h)$-smooth on $I$, for $h \in ]0; h_0]$. Then

$$\int_{\mathbb{R}} \left\| \langle x\rangle^{-s} \theta\big(\hat{P}(h)\big) U(t) \right\|^2 dt \ \leq \ C_s \,, \tag{4.1}$$

where $U(t) := \exp(-ih^{-1}t\hat{P}(h))$, $\theta \in C_0^\infty(I; \mathbb{R})$ satisfies $0 \leq \theta \leq 1$ and equals 1 near $\lambda$, and $C_s > 0$ only depends on the resolvent estimates, $s$ and $I$. Now, for $B = \langle x\rangle^{-2s}\mathrm{I}_m$, we want an approximation of $U(t)^* B_h^w U(t)$ of the form $(F^t(B))_h^w$ where $F^t$ should map symbols to symbols and $F^0$ be the identity. By $\tilde{O}_t(h)$, we denote a bounded operator on $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$, the norm of which is $O_t(h)$. We write, using (3.7),

$$\begin{aligned} U(t)^* B_h^w U(t) \, - \, \Big(F^t(B)\Big)_h^w \ &= \ - \int_0^t (d/dr)\Big( U(t-r)^* \, \Big(F^r(B)\Big)_h^w U(t-r)\Big)\, dr \\ &= \ - \int_0^t U(t-r)^* \, ih^{-1}\Big([P, F^r(B)]\Big)_h^w U(t-r)\, dr \, + \, \tilde{O}_t(h) \\ &\quad - \int_0^t U(t-r)^* \, \Big(dF^r(B)/dr \, - \, \{P, F^r(B)\}\Big)_h^w U(t-r)\, dr \,. \end{aligned} \tag{4.2}$$

In the scalar case (see [W]), one can choose $F^t(B)$ to be $B \circ \phi^t$, where $\phi^t$ is the classical Hamiltonian flow, and one has a really good approximation for $U(t)^* B_h^w U(t)$ (Egorov's theorem), since the r.h.s of (4.2) is some $\tilde{O}_t(h)$. Then, it is easy to translate (4.1) into some integrability property of $(F^t(B))_h^w$, which leads to the non-trapping condition on $\phi^t$ at energy $\lambda$ by use of coherent states microlocalized on the energy shell of energy $\lambda$ (see [W] or the arguments below).

We come to the matricial case introduced in Subsection 2.1 with empty relevant eigenvalues crossing at energy $\lambda$ (cf. Definition 2.2). In particular, we have the situation described at the end of Definition 2.4. Let $N$ be the number of relevant eigenvalues at energy $\lambda$. Assume for a while that there is no relevant eigenvalues crossing at energy $\lambda$ "at infinity". This means that we can find $\delta, R > 0$ such that any two different relevant eigenvalues at energy $\lambda$, $\lambda_j$ and $\lambda_k$, satisfy the property $(|x| \geq R \Longrightarrow |\lambda_j(x) - \lambda_k(x)| \geq \delta)$.

It turns out that we only need some control of the l.h.s of (4.2) localized in energy, that is after multiplying on both sides by some $\theta(\hat{P}(h))$. By the $h$-pseudodifferential calculus, the principal symbol of $\theta(\hat{P}(h))([P, F^t(B)])_h^w\theta(\hat{P}(h))$ is given by $\theta(P)[P, F^t(B)]\theta(P)$. By Proposition 2.3, it is zero if the support of $\theta$ is close enough to $\lambda$. In particular, we get a localized, weak version of Egorov's theorem. Indeed, we define on $E^*(\lambda, \epsilon)$ with small enough $\epsilon > 0$, for

$$A \;=\; \sum_{j=1}^{N} a_j \Pi_j\,, \qquad F^t(A) \;:=\; \sum_{j=1}^{N} \Big(a_j \circ \phi_j^t\Big)\Pi_j\,. \tag{4.3}$$

Thanks to Proposition 2.5, $\sum_{j=1}^{N} \Pi_j \{P, F^t(A)\}\Pi_j \;=\; dF^t(A)/dt$. Since the crossing is empty, $\theta(p_j)\theta(p_k) = 0$ for $k \neq j$, if the support of $\theta$ is close enough to $\lambda$. In particular,

$$\theta(P)\Bigg(\sum_{j,k=1, j\neq k}^{N} \Pi_j\{P, F^t(A)\}\Pi_k\Bigg)\theta(P) \;=\; 0 \tag{4.4}$$

Making use of (3.3) and (4.2), the previous arguments show that

$$\Big\|\, \theta\Big(\hat{P}(h)\Big)\, U(t)^* B_h^w U(t)\, \theta\Big(\hat{P}(h)\Big) \;-\; \theta\Big(\hat{P}(h)\Big)\, \Big(F^t(B)\Big)_h^w\, \theta\Big(\hat{P}(h)\Big) \,\Big\| \;=\; O_t(h)\,. \tag{4.5}$$

Thus we get, for any $T > 0$,

$$\begin{aligned}\int_{-T}^{T} \theta\Big(\hat{P}(h)\Big)\, U(t)^* B_h^w U(t)\, \theta\Big(\hat{P}(h)\Big)\, dt \;&=\; \int_{-T}^{T} \theta\Big(\hat{P}(h)\Big)\Big(F^t(B)\Big)_h^w \theta\Big(\hat{P}(h)\Big)\, dt \\ &\quad + \tilde{O}_T(h)\,. \end{aligned}\tag{4.6}$$

Now, we introduce the coherent states operator microlocalized near $\mathrm{x}_0^* = (x_0, \xi_0) \in T^*\mathbb{R}^d$. It is the unitary operator on $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^m)$ given by

$$c(\mathrm{x}_0^*) := \exp\Big(ih^{-1/2}(x \cdot x_0 - \xi_0 \cdot D_x)\Big)\, \mathrm{I}_m\,, \tag{4.7}$$

(cf. [R, J2]). For any $S \in \Sigma_{r,t}$ with $r, t \leq 0$ (cf. 3.1), $c(\mathrm{x}_0^*)^* S_h^w c(\mathrm{x}_0^*) \;=\; S(\mathrm{x}_0^*) + \tilde{O}_S(h)$. Requiring that $\theta(\lambda) = 1$, we obtain, for $\mathrm{x}_0^* \in E(\lambda)$,

$$\int_{-T}^{T} c(\mathrm{x}_0^*)^* \theta\Big(\hat{P}(h)\Big)\Big(F^t(B)\Big)_h^w \theta\Big(\hat{P}(h)\Big) c(\mathrm{x}_0^*)\, dt \;=\; \int_{-T}^{T} F^t(B)(\mathrm{x}_0^*)\, dt \;+\; \tilde{O}_T(h)\,.$$

Notice that $F^t(B)(\mathrm{x}_0^*) = \sum_{j=1}^N \langle \pi_x \phi_j^t(\mathrm{x}_0^*)\rangle^{-2s}\Pi_j(x_0)$, if $\pi_x : T^*\mathbb{R}^d \longrightarrow \mathbb{R}^d$ is defined by $\pi_x(q,p) = q$. Letting $h$ tend to zero and using (4.1) and (4.6), we conclude that, there exists some $C > 0$ such that, for all $j \in \{1, \cdots, N\}$, all $\mathrm{x}_0^* \in p_j^{-1}(\lambda)$, and all $T > 0$,

$$\int_{-T}^{T} \Big\langle \pi_x \phi_j^t(\mathrm{x}_0^*) \Big\rangle^{-s} dt \;\leq\; C\,. \tag{4.8}$$

As in [W], this implies that $\lambda$ is non-trapping for each Hamiltonian flow $\phi_j^t$. Now, let us remove the assumption at infinity we made above. Let $\chi \in C_0^\infty(\mathbb{R};\mathbb{R})$ such that $0 \leq \chi \leq 1$ and $\chi(0) = 1$. For $R > 0$, set $\chi_R(x) = \chi(x/R)$. We can follow the previous lines with $B$ replaced by $\chi_R B$. This leads to

$$\int_{-T}^{T} \chi_R\Big(\pi_x \phi_j^t(\mathrm{x}_0^*)\Big) \Big\langle \pi_x \phi_j^t(\mathrm{x}_0^*) \Big\rangle^{-s} dt \;\leq\; C\,.$$

Taking the supremum over $R \geq 1$, we derive (4.8), since $C$ is independent of $R$. Thus, we have proved the

**Proposition 4.1.** *Consider the general case* (*cf. Subsection 2.1*) *and take* $\lambda \in \mathbb{R}$. *Assume that the relevant crossing at energy* $\lambda$ (*cf. Definition 2.2* ) *is empty. Assume further that there exist* $h_0 > 0$ *and an open interval* $I$ *about* $\lambda$ *such that the boundary values* $R(\mu + i0)$ *and* $R(\mu - i0)$ *exist on* $I$, *for* $h \in ]0; h_0]$ *and all* $s > 1/2$ (cf. (1.3)). *If, for some* $s > 1/2$, *for* $h$ *small enough, and near* $\lambda$, *the resolvent estimate* (1.4) *and the same estimate for* $R(\mu - i0)$ *hold true, then, for all relevant eigenvalues* $p_j$ *at energy* $\lambda$, $\lambda$ *is a non-trapping energy for* $p_j$ (*cf.* (1.5) *and Definition 2.4*).

**Remark 4.2.** *In the proof of Proposition 4.1, we can derive* (*4.5*) *from the matricial Egorov theorems of [BG, BN], if there is no crossing at infinity* (*cf. Definition 2.8*).

## 4.2 A Gronwall type argument.

In this subsection, we adapt the strategy considered in Subsection 4.1 to prove Theorem 1.3. Because of the eigenvalues crossing, the first term on the r.h.s of (4.2) does not vanish at all by energy localization. Therefore, we need to modify the strategy. It is reasonable to solve the differential equation $dF^t(B)/dt = \{P, F^t(B)\}$ but it is not clear that the solution commute with $P$ near the considered energy shell. So we do not know how to generalize the approximation (4.5), if it is possible. We choose to avoid this question and to require that the first term on the r.h.s of (4.2) vanishes. Since we consider a model of codimension 1 crossing (cf. Definition 2.6), the functions in (4.3) are nice symbols satisfying the requirement. But we do not expect anymore to solve $dF^t(B)/dt = \{P, F^t(B)\}$ near the energy shell (see (4.4)). Therefore, up to some $O_t(h)$, there is a term left on the r.h.s of (4.2). Using the structure condition at infinity (cf. Definition 2.9), we roughly control it by $\|(F^t(B))_h^w\|$ and get, by a Gronwall type argument, the expected time integrability of $(F^t(B))_h^w$ (cf. Proposition 4.3). Then, we conclude as in Subsection 4.1.
We point out that, if there is a "crossing at infinity" (cf. Definition 2.8), we do need the structure condition at infinity (cf. Definition 2.9) and we do use Proposition 1.6 to perform the Gronwall type argument. Otherwise, a simplier proof works.

**Proposition 4.3.** *Consider the model of codimension* 1 *crossings* (*see Definition 2.6*) *that satisfies the structure condition at infinity* (*see Definition 2.9*). *Let* $\lambda > \|M_\infty\|_m$. *Assume that* (4.1) *holds true for some* $s \in ]1/2;(1+\rho)/2]$ ($\rho$ *appears in* (2.1)) *if there is a "crossing at infinity"* (*see Definition 2.8*) *and for some* $s > 1/2$, *otherwise. Let* $B = \langle x\rangle^{-2s}\mathrm{I}_m$ *and define* $F^t(B)$ *as in (4.3). Let* $\chi$, $\chi(\cdot;\cdot,\cdot)$ *be as in (2.7) with* $\chi = 1$ *on* $[-1/2;1/2]$ *and* $0 \le \chi \le 1$. *Then, we can find* $C > 0$ *such that, for all* $T > 0$, *there exists* $\epsilon > 0$, *such that, for* $h$ *small enough,*

$$\int_{-T}^{T} \Big\|\chi\Big(\hat{P}(h);\lambda,\epsilon\Big)\,\Big(F^t(B)\Big)_h^w\,\chi\Big(\hat{P}(h);\lambda,\epsilon\Big)\Big\|\,dt \;\le\; C\,.$$

**Proof:** First, we write a localized version of (4.2), using (3.3) and $[P, F^r(B)] = 0$.

$$\begin{aligned}
&\chi\Big(\hat{P}(h);\lambda,\epsilon\Big)\Big(U(t)^*B^wU(t)\;-\;\Big(F^t(B)\Big)_h^w\Big)\chi\Big(\hat{P}(h);\lambda,\epsilon\Big)\\
&=\;-\int_0^t U(t-r)^*\,\Big(\chi(P;\lambda,\epsilon)\Big)_h^w\,\Big(dF^r(B)/dr\;-\;\Big\{P,F^r(B)\Big\}\Big)_h^w\cdot\\
&\qquad\Big(\chi(P;\lambda,\epsilon)\Big)_h^w\,U(t-r)\,dr\;+\;\tilde{O}_t(h)\\
&=\;-\int_0^t U(t-r)^*\,\Big(\chi(P;\lambda,\epsilon)\Big)_h^w\,\Big(\sum_{j,k=1,j\neq k}^{N}\Pi_j\{P,F^r(B)\}\Pi_k\Big)_h^w\cdot\\
&\qquad\Big(\chi(P;\lambda,\epsilon)\Big)_h^w\,U(t-r)\,dr\;+\;\tilde{O}_t(h)\,,
\end{aligned}\tag{4.9}$$

since $\sum_{j=1}^N\Pi_j\{P,F^t(B)\}\Pi_j = dF^t(B)/dt$. Recall that $\tilde{O}_t(h)$ denote a bounded operator on $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^m)$, the norm of which is $O_t(h)$. In view of Definitions 2.6 and 2.9, we introduce a partition of unity localized on $E^*(\lambda,\epsilon_0)$ (cf. (2.2)), for some $\epsilon_0 > 0$. Let $\psi,\psi_0,\psi_1,\cdots,\psi_p \in C^\infty(T^*\mathbb{R}^d;\mathbb{R})$ with $\psi+\sum_{q=0}^p\psi_q = 1$ on $E^*(\lambda,\epsilon_0)$ and satisfying the following conditions. $\psi_0$ is supported in $\{(x,\xi)\in T^*\mathbb{R}^d; |x| > R\}$, for some $R > 0$, where $M$ has the structure mentioned in Definition 2.9. In particular, (2.11) and (2.12) hold true. The other functions are compactly supported and $\psi$ is supported away from $\mathcal{C}^*(\lambda)$ (see (2.6)). For each $k\in\{1,\cdots,p\}$, $M$ has the matricial structure described in Definition 2.6 on the support of $\psi_k$. If there is "no crossing at infinity" (cf. Definition 2.8), we remove the function $\psi_0$ and take $\psi$ supported away from $\mathcal{C}^*(\lambda)$ and at "infinity".
We insert $1 = \psi^w+\sum_{q=0}^p\psi_q^w$ into (4.9), between the two energy localizations $(\chi(P;\lambda,\epsilon))_h^w$. The contribution of $\psi$ is $\tilde{O}_t(h)$, since (4.4) holds true on the support of $\psi$. Let $q\in\{1,\cdots,p\}$ and consider the contribution of $\psi_q$. As in the proof of Proposition 3.6, we use the fact that $\chi(P(1-\Pi_I);\lambda,\epsilon)$ and $\psi_q$ have disjoint supports, to see that the contribution of $\psi_q$ in (4.9)

$$\begin{aligned}
&=\;-\int_0^t U(t-r)^*\,\Big(\chi(P;\lambda,\epsilon)\Big)_h^w\,\Big(\sum_{j,k\in I,j\neq k}\Pi_j\Big\{P_I,\Big(F^r(B)\Big)_I\Big\}\Pi_k\,\psi_q\Big)_h^w\\
&\qquad\cdot\Big(\chi(P;\lambda,\epsilon)\Big)_h^w\,U(t-r)\,dr\;+\;\tilde{O}_t(h)\,.
\end{aligned}\tag{4.10}$$

As in the proof of Proposition 3.7, writing $F^r(B) = \sum_{l=1}^N a_l^r\Pi_l$ and using Proposition 2.5,

$$\sum_{j,k\in I,j\neq k}\Pi_j\Big\{P_I,\Big(F^r(B)\Big)_I\Big\}\Pi_k\,\psi_q\;=\;\sum_{j,k\in I,j\neq k}\Pi_j\Big(Q_1(r)\;+\;\tau Q_2(r)\Big)\Pi_k\,\psi_q\,,$$

where $\|(Q_2(r)\psi_q)_h^w\| = O_r(1)$ and $Q_1(r) = \sum_{l\in I} a_l^r(2\xi\cdot\nabla\Pi_l)$ on the support of $\psi_q$. But

$$\begin{aligned}
&\Big(\chi(P;\lambda,\epsilon)\Big)_h^w \Big(\sum_{j,k\in I,\, j\neq k} \Pi_j\,\tau\,Q_2(r)\,\Pi_k\,\psi_q\Big)_h^w \Big(\chi(P;\lambda,\epsilon)\Big)_h^w \\
=\ &\Big(\chi(P;\lambda,\epsilon)\Big)_h^w \tau\chi(\tau;0,\epsilon)\Big(\sum_{j,k\in I,\, j\neq k} \Pi_j\,Q_2(r)\,\Pi_k\,\psi_q\Big)_h^w \Big(\chi(P;\lambda,\epsilon)\Big)_h^w + \tilde{O}_r(h)\,,
\end{aligned}$$

thus the contribution of $Q_2$ in (4.10) is $\tilde{O}_t(\epsilon)+\tilde{O}_t(h)$. Since $\psi_q$ is compactly supported and $F^r(B)\Pi_l = a_l\Pi_l$, the contribution of $Q_1$ in (4.10) is

$$\begin{aligned}
=\ -&\int_0^t U(t-r)^*\,\chi\Big(\hat{P}(h);\lambda,2\epsilon\Big)\,\langle x\rangle^{-s}\,\chi\Big(\hat{P}(h);\lambda,\epsilon\Big)\Big(F^r(B)\Big)_h^w\chi\Big(\hat{P}(h);\lambda,\epsilon\Big) \qquad (4.11)\\
&\cdot\Big(\langle x\rangle^{2s}\sum_{\substack{j,k,l\in I\\ j\neq k}}\Pi_j\Big(2\xi\cdot\nabla\Pi_l\Big)\Pi_k\,\psi_q\Big)_h^w\langle x\rangle^{-s}\,\chi\Big(\hat{P}(h);\lambda,2\epsilon\Big)\,U(t-r)\,dr + \tilde{O}_t(h)\,,
\end{aligned}$$

where the factor containing $\langle x\rangle^{2s}$ is uniformly bounded w.r.t. $h$. Here we used $\chi(\cdot;\lambda,\epsilon) = \chi(\cdot;\lambda,2\epsilon)\chi(\cdot;\lambda,\epsilon)$. Although the support of $\psi_0$ is not compact, the same computation works for $q=0$ thanks to (2.11) and (2.12) (the set $I$ must be replaced by $I_\infty$ defined in (2.10)). The factor containing $\langle x\rangle^{2s}$ in (4.11) for $q=0$ is also uniformly bounded if $s\in]1/2;(1+\rho)/2]$.
Let $T>0$. Let $\hat{\chi} = \chi(\hat{P}(h);\lambda,\epsilon)$ and $\hat{\chi_0} = \chi(\hat{P}(h);\lambda,2\epsilon)$, for short. Putting all together, we arrive at, for some $Q\in\Sigma_{0,0}$,

$$\begin{aligned}
\int_0^T \hat{\chi}\Big(F^r(B)\Big)_h^w\hat{\chi}\,dt\ &=\ \int_0^T \hat{\chi}U(t)^*B_h^wU(t)\hat{\chi}\,dt + \tilde{O}_T(\epsilon) + \tilde{O}_T(h) \qquad (4.12)\\
&+\int_0^T U(r)^*\hat{\chi_0}\langle x\rangle^{-s}\Big(\int_0^t \hat{\chi}\Big(F^{t'}(B)\Big)_h^w\hat{\chi}\,dt'\Big)\,Q_h^w\,\langle x\rangle^{-s}\,\hat{\chi_0}U(r)\,dr\,.
\end{aligned}$$

Let $q>0$ such that $\|Q_h^w\|\leq q$, for small enough $h$. For $\epsilon$ and $h$ small enough, the sum of the three first terms on the r.h.s of (4.12) is in norm smaller than $2C_s$, by (4.1). Using again (4.1), we can apply Gronwall's lemma (see [DG], for instance) to the function $T\mapsto\int_0^T\|\hat{\chi}(F^r(B))_h^w\hat{\chi}\|\,dt$. Thus, this function is bounded by $2C_s\exp(qC_s)$. Similarly, we can control the integral over $[-T;0]$. □

**Proof of Theorem 1.3:** By assumption, the property ((1.3) and (1.4)) at energy $\lambda > \|M_\infty\|_m$ holds true for some $s>1/2$. If there is a "crossing at infinity" (cf. Definition 2.8), we can assume, by Proposition 1.6, that $s\in]1/2;(1+\rho)/2]$, where $\rho>0$ measures the decay of $M$ at infinity (cf. (2.1)). Now, we apply Proposition 1.7. Thus, for this $s$, the same property applied to $R(\lambda-i\epsilon)$, instead of $R(\lambda+i\epsilon)$, holds true. Denoting by $U(t) := \exp(-ih^{-1}t\hat{P}(h))$ the propagator of $\hat{P}(h)$, this allows us to use Kato's local smoothness to derive (4.1). Now, we use Proposition 4.3. Thus we can find some $C>0$ such that, for all $T>0$, there is some $\theta\in C_0^\infty(\mathbb{R};\mathbb{R})$, with $\theta=1$ near $\lambda$, such that, for $h$ small enough,

$$\int_{-T}^T \Big\|\theta\Big(\hat{P}(h)\Big)\,U(t)^*\Big(\sum_{j=1}^N\Big\langle\pi_x\phi_j^t(\cdot)\Big\rangle^{-2s}\Pi_j\Big)_h^w U(t)\,\theta\Big(\hat{P}(h)\Big)\Big\|\,dt\ \leq\ C\,.$$

Here, $\pi_x \phi_j^t$ denotes the space component of the flow $\phi_j^t$. Using coherent states operators (4.7), we translate this integrability into (4.8), as in Subsection 4.1, yielding the non-trapping condition, as in [W]. □

# 5 Previous results revisited.

In this last section, we want to compare the present results with those of [J1, J4]. In some sense, we complete and generalize the later ones. Furthermore, we show that the method developped in Section 3 can be used to derive a new proof of them.
Let us first explain what kind of improvement of results we have. In [J1], the relevant eigenvalues crossing at considered energy is empty and the property ((1.3) and (1.4)) is deduced from a non-trapping condition on the flows of the eigenvalues of the symbol. In [J4], we consider a codimension 1 crossing and we essentially require that a global eigenvalue only crosses another one (see Definition 2.6). We assume further that the variation of eigenspaces of $M(x)$ is small enough. Then we prove the property ((1.3) and (1.4)) under the previous non-trapping condition.
These results are completed here in the sense that we show that this non-trapping condition is necessary to have the semiclassical resolvent estimates (see Theorems 1.3 and 1.8). In the case treated in [J1], we even have a simplier proof (see the proof of Proposition 4.1). Since we assume an invariance condition on $M$ in Theorem 1.2, we cannot deduce from it the results in [J4]. But, we think that this invariance condition is more satisfactory than the previous, vague requirement on the variation of the eigenspaces. Furthermore, we are able here to consider much more general eigenvalues crossings.
Another interesting feature is that we do not loose anything if we adopt the method developped in Section 3. Let us show how it gives a new proof of the previously mentioned results in [J1, J4]. It turns out that these results can be derived from the following theorem, that we proved in [J4] with the semiclassical Mourre theory.

**Theorem 5.1. [J4]** *Consider the general model* (*see Subsection 2.1* ) *and let* $\lambda > \|M_\infty\|_m$*. Assume that we can find some* $c > 0$ *and some function* $A \in C^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$ *such that* $A(x, \xi) = (x \cdot \xi)\mathrm{I}_m$ *for* $|x|$ *large enough and such that, near* $E^*(\lambda)$ (*cf.* (2.3)), $[P, A]$ *vanishes and* $\{P, A\} \geq c\mathrm{I}_m$ (*cf.* (3.8)). *Then, for all* $s > 1/2$*, the property* ((1.3) *and* (1.4)) *holds true.*

Here we shall show the following stronger result.

**Proposition 5.2.** *Consider the general model* (*see Subsection 2.1* ) *and let* $\lambda > \|M_\infty\|_m$*. Assume that, for any* $R > 0$*, we can find some* $c > 0$ *and some* $A \in C^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$ *such that* $[P, A]$ *vanishes near* $\mathcal{C}^*(\lambda) \cap \{(x, \xi) \in T^*\mathbb{R}^d; |x| \leq R\}$ (*cf.* (2.6)) *and such that* $\{P, A\} \geq c\mathrm{I}_m$ *near* $E^*(\lambda) \cap \{(x, \xi) \in T^*\mathbb{R}^d; |x| \leq R\}$ (*cf.* (2.3)). *Then, for all* $s > 1/2$*, the property* ((1.3) *and* (1.4)) *holds true.*

**Proof:** As in Section 3, we assume that the property ((1.3) and (1.4)) fails at energy $\lambda$ for some $s > 1/2$. Thus Hypothesis 1 holds true. By Proposition 3.4, $\mu$ is nonzero and there exists some $R_0 > 0$ such that $\mu$ is supported in $E^*(\lambda) \cap \{(x,\xi) \in T^*\mathbb{R}^d; |x| \leq R_0\}$. Let $A \in C^\infty(T^*\mathbb{R}^d; \mathcal{M}_m(\mathbb{C}))$ and $c > 0$ given by the assumption for $R = R_0$. By Proposition 3.5, $\nu$ is supported in $\mathcal{C}^*(\lambda) \cap \{(x,\xi) \in T^*\mathbb{R}^d; |x| \leq R_0\}$ and $0 = \nu(A) = \mu(\{P, A\}) \geq c\mu(\mathrm{I}_m)$. This contradicts $\mu \neq 0$. $\square$

# References

[Ba] A.Balazard-Konlein: *Calcul fonctionnel pour des opérateurs h-admissibles à symbole opérateur et applications.* PhD Thesis, Université de Nantes, 1985, and also C.R. Acad. Sci. Paris, Série I 301, 903-906, 1985.

[BG] J.Bolte, R.Glaser: *A semiclassical Egorov theorem and quantum ergodicity for matrix valued operators.* preprint, arXiv:math-ph/0204018, 2002.

[BN] R.Brummelhuis, J.Nourrigat: *Scattering amplitude for Dirac operators.* Comm. PDE 24 (1 & 2), 377-394 (1999).

[B] N.Burq: *Semiclassical estimates for the resolvent in non trapping geometries.* Int. Math. Res. Notices 2002, 5, 221-241.

[CLP] Y.Colin de Verdière, M.Lombardi, J.Pollet: *The microlocal Landau-Zener formula.* Ann. Inst. Henri Poincaré, Phys. Théor. 71, 1, 95-127 (1999).

[CFKS] H.L.Cycon, R.Froese, W.Kirsch, B.Simon: *Schrödinger operators with application to quantum mechanics and global geometry.* Springer-Verlag 1987.

[DG] J.Dereziński, C.Gérard : *Scattering theory of classical and quantum N-particle systems.* Springer-Verlag 1997.

[FG] C.Fermanian, P.Gérard : *Mesures semi-classiques et croisement de modes.* Bull. Soc. math. France 130 (1), 2002, p. 123-168.

[FH] R.Froese, I.Herbst : *Exponential bounds and absence of positive eigenvalues for N-body Schrödinger operators.* Comm. Math. Phys. 87, 3, p. 429-447 (1982/1983).

[GM] C.Gérard, A.Martinez : *Principe d'absorption limite pour des opérateurs de Schrödinger à longue portée.* C.R. Acad. Sci. Paris, Série I 306, 121-123, 1988.

[GL] P.Gérard, E.Leichtnam : *Ergodic properties of eigenfunctions for the Dirichlet problem.* Duke Math. J. 71, 2, 1993.

[H] G.A.Hagedorn : *Molecular propagation through electron energy level crossings.* Memoirs AMS 536, vol. 111, 1994.

[J1] Th.Jecko : *Estimations de la résolvante pour une molécule diatomique dans l'approximation de Born-Oppenheimer.* Comm. Math. Phys. 195, 3, 585-612, 1998.

[J2] Th.Jecko : *Classical limit of elastic scattering operator of a diatomic molecule in the Born-Oppenheimer approximation.* Ann. Inst. Henri Poincaré, Physique théorique, 69, 1, 1998, p. 83-131.

[J3] Th.Jecko : *Approximation de Born-Oppenheimer de sections efficaces totales diatomiques.* Asympt. Anal. 24 (2000), p. 1-35.

[J4] Th.Jecko : *Semiclassical resolvent estimates for Schrödinger matrix operators with eigenvalues crossing.*, Math. Nachr. 257, 36-54 (2003).

[J5] Th.Jecko : *From classical to semiclassical non-trapping behaviour.* C.R. Acad. Sci. Paris, Série I, in press. Preprint available on 02-476 at:
$http//rene.ma.utexas.edu/mp\_arc/index - 02.html\#end$

[Ka] U.Karlsson : *Semi-classical approximations of quantum mechanical problems.* PhD thesis, KTH, Stockholm 2002.

[K] T.Kato : *Pertubation theory for linear operators.* Springer-Verlag 1995.

[KMW] M.Klein, A.Martinez, X.P.Wang : *On the Born-Oppenheimer approximation of wave operators in molecular Scattering.* Comm. Math. Phys 152, 73-95 (1993).

[M] E.Mourre : *Absence of singular continuous spectrum for certain self-adjoint operators.* Comm. Math. Phys. 78, 391-408, 1981.

[Né] L.Nédélec : *Resonances for matrix Schrödinger operators.* Duke Math. J. 106, 2, 209-236 (2001).

[N] F. Nier: *A semiclassical picture of quantum scattering.* Ann. Sci. École Norm. Sup. (4) 29 (1996), 2, 149-183.

[RS2] M.Reed, B.Simon : *Methods of Modern Mathematical Physics, Tome II : Fourier Analysis, Self-adjointness.* Academic Press 1979.

[RS4] M.Reed, B.Simon : *Methods of Modern Mathematical Physics, Tome IV : Analysis of operators.* Academic Press 1979.

[R] D.Robert : *Autour de l'approximation semi-classique.* Birkhäuser 1987.

[RT] D.Robert, H.Tamura : *Semiclassical estimates for resolvents and asymptotics for total cross-section.* Ann. IHP 46, 1987, pp. 415-442.

[S] V.Sordoni : *Reduction scheme for semiclassical operator-valued Schrödinger type equation and application to scattering.* Comm. PDE, 28, 7 & 8, pp. 1221-1236, 2003.

[VZ] A. Vasy, M. Zworski: *Semiclassical estimates in asymptotically euclidean scattering.* Comm. Math. Phys. 212, 1, pp. 205-217 (2000).

[W] X.P.Wang : *Semiclassical resolvent estimates for $N$-body Schrödinger operators.* J. Funct. Anal. 97, 466-483 (1991).